The Minimal Balancing Multiplier and the Riemann Zeta Function at Positive Integers

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Abstract

We study the distribution of central conjugate divisors across quadratic shells, defined as the intervals between consecutive squares. We investigate two dual sequences: the shell discrepancy $c(n)$ (A381054), which partitions the positive integers into balanced integers (zero discrepancy) and eccentric integers (positive discrepancy), and the minimal balancing multiplier $\beta(n)$ (A382286), the least positive multiplier whose product with a given integer is balanced. We determine the cumulative asymptotic growth of the shell counting functions for balanced (A382306) and eccentric (A383115) integers, establish exact characterizations of primes and odd prime powers via $\beta(n)$, and prove that Apéry's constant $\zeta(3)$ governs both the asymptotic density of the square-balanced set $\mathcal{L}_2$ (A401528) and the summatory function of $\beta(n)$. Finally, we prove that the s-th moments of the shell-normalized balancing multiplier generate all odd zeta values $\zeta(2s+1)$, while extending $\beta(n)$ to the two-parameter foliated hierarchy $\beta_m^{(r)}(n)$ across r-th power shells shows that the r-th power multiplier sets $\mathcal{L}_m^{(r)}$ have asymptotic density $\zeta(r+1)$ relative to $\pi(x)$, separating even-degree foliations (odd zeta values $\zeta(2s+1)$) from odd-degree foliations (even zeta values $\zeta(2s)$).

1 Introduction

The multiplicative structure of a positive integer $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ can be investigated at two complementary levels [9, 14, 15]. At the atomic level—its prime factorization—one can classify integers either by restricting the exponents α_i, as in squarefree integers ($\alpha_i = 1$, of asymptotic density $1/\zeta(2) = 6/\pi^2$), b-free integers ($\alpha_i < b$, of asymptotic density $1/\zeta(b)$), and powerful integers ($\alpha_i \geq 2$), or by bounding the size of the prime factors p_i, as in y-smooth (or y-friable) integers satisfying $\max_{p|n} p \leq y$ [9, 19].

At the molecular level—the divisors—one can study their global distribution, and in particular the proximity of the conjugate divisor pairs (a, b) satisfying $ab = n$ and $1 \leq a \leq \sqrt{n} \leq b$ [14]. Among all conjugate pairs, the central divisors

$$d_1(n) = \max\{d \mid n : d \leq \sqrt{n}\} \quad (\text{A033676}), \qquad d_2(n) = \frac{n}{d_1(n)} \quad (\text{A033677}) \tag{1}$$

minimize the classical difference $b - a$. Previous investigations of central divisor proximity have focused either on fixing the difference $d_2(n) - d_1(n) = k$—which produces squares

(A000290) at $k = 0$, pronic numbers (A002378) at $k = 1$, and the factor-difference sets of Erdős and Rosenfeld [10] of growth $O(x^{1/2})$—or on bounding the multiplicative ratio $d_2(n)/d_1(n)$ to study divisors in short intervals, as in the work of Hall and Tenenbaum [14, 19] and Ford [12].

In this paper, we introduce a hierarchy of scales for measuring central divisor proximity by partitioning the positive integers into shells composed of the discrete intervals between consecutive r-th powers ($r \geq 2$), via the floor of the r-th root (A000196 for $r = 2$), $\lfloor n^{1/r} \rfloor$ [5, 13]. Measuring the shell distance between $d_1(n)$ and $d_2(n)$ gives the shell discrepancy $c^{(r)}(n) = \lfloor d_2(n)^{1/r} \rfloor - \lfloor d_1(n)^{1/r} \rfloor$, whose linear case $r = 1$ recovers the factor-difference sets of Erdős and Rosenfeld [10]. In the quadratic case $r = 2$ (where we write $c(n) = c^{(2)}(n)$, A381054, for simplicity of notation), this measurement partitions the positive integers into balanced integers ($c(n) = 0$, counted per shell by A382306) and eccentric integers ($c(n) \geq 1$, with $c(n) = 1$ counted per shell by A383115), both of which, as we will demonstrate, have cumulative growth $O(x^{3/4})$. Consequently, we investigate the minimal balancing multiplier $\beta(n) = \min\{k \geq 1 : c(kn) = 0\}$ (A382286), its square-balanced set $\mathcal{L}_2$ (A401528)—$\beta(n)$ is a square—and its two-parameter extension $\beta_m^{(r)}(n)$ across r-th power shells, showing that while b-free integers at the prime-factorization level are governed by the reciprocal zeta values $1/\zeta(b)$, the moments and r-th power-balanced sets of the minimal balancing multiplier generate the positive integer values $\zeta(r+1)$ of the Zeta function directly relative to $\pi(x)$.

Definition 1. For each integer $k \geq 1$, the k-th *quadratic shell* S_k is the discrete interval

$$S_k = \{n \in \mathbb{N} : \lfloor \sqrt{n} \rfloor = k\} = [k^2, k^2 + 2k]. \tag{2}$$

Each shell S_k has cardinality $|S_k| = 2k + 1$ and midpoint $k(k+1) = k^2 + k$.

Comparing the shell indices of $d_1(n)$ and $d_2(n)$ defines two dual arithmetic sequences in the On-Line Encyclopedia of Integer Sequences [17].

Definition 2. Let $n \geq 1$ be an integer with central conjugate divisors $d_1(n) \leq d_2(n)$.

1. The *shell discrepancy* of n (A381054) is the non-negative integer

$$c(n) = \lfloor \sqrt{d_2(n)} \rfloor - \lfloor \sqrt{d_1(n)} \rfloor. \tag{3}$$

   For each integer $m \geq 0$, we let $\mathcal{C}_m = \{n \geq 1 : c(n) = m\}$ denote the m-discrepancy set. Integers in $\mathcal{B} = \mathcal{C}_0$ are called *balanced*, and integers with $c(n) \geq 1$ are called *eccentric*.

2. For any integer $n \geq 1$, the set of balancing multipliers $\mathcal{M}(n) = \{k \geq 1 : c(kn) = 0\}$ is non-empty, since $k = n$ yields the square n^2 with $d_1(n^2) = d_2(n^2) = n$ and $c(n^2) = 0$. Moreover, $\mathcal{M}(n)$ is infinite: indeed, for every integer $m \geq 1$, the multiple $k' = nm^2$ produces the square $k'n = (nm)^2$ satisfying $c(k'n) = 0$. By the well-ordering principle of the positive integers, $\mathcal{M}(n)$ possesses a unique least element, called the *minimal balancing multiplier* of n (A382286):

$$\beta(n) = \min\{k \geq 1 : c(kn) = 0\}. \tag{4}$$

Table 1 displays the values of $d_1(n)$, $d_2(n)$, $c(n)$, and $\beta(n)$ for $1 \leq n \leq 20$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$d_1(n)$	1	1	1	2	1	2	1	2	3	2	1	3	1	2	3	4	1	3	1	4
$d_2(n)$	1	2	3	2	5	3	7	4	3	5	11	4	13	7	5	4	17	6	19	5
$\lfloor\sqrt{d_1(n)}\rfloor$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	2
$\lfloor\sqrt{d_2(n)}\rfloor$	1	1	1	1	2	1	2	2	1	2	3	2	3	2	2	2	4	2	4	2
$c(n)$	0	0	0	0	1	0	1	1	0	1	2	1	2	1	1	0	3	1	3	0
$\beta(n)$	1	1	1	1	4	1	4	2	1	2	9	2	9	2	2	1	16	2	16	1

Table 1: Values of $d_1(n)$, $d_2(n)$, $c(n)$ (A381054), and $\beta(n)$ (A382286) for $1 \le n \le 20$.

2 Centrality of Same-Shell Factor Pairs and Asymptotics of $h_m(k)$

A foundational property of the quadratic shells S_k is that any two numbers in that shell are necessarily the central conjugate divisors of their product.

Lemma 3. *Let $n = ab$ with $1 \le a \le b$. If $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor = k$, then $(a, b) = (d_1(n), d_2(n))$ and $c(n) = 0$.*

Proof. Let $a, b \in S_k$, so that $k \le \sqrt{a} \le \sqrt{b} < k + 1$. Then $0 \le \sqrt{b} - \sqrt{a} < 1$. For the central divisor pair $(c, d) = (d_1(n), d_2(n))$, we have $a \le c \le \sqrt{n} \le d \le b$, and the arithmetic-geometric mean inequality gives

$$0 \le (a + b) - (c + d) \le (a + b) - 2\sqrt{ab} = (\sqrt{b} - \sqrt{a})^2 < 1. \tag{5}$$

Because $(a+b)-(c+d)$ is a non-negative integer strictly less than 1, we must have $a+b = c+d$. Since $ab = cd = n$ and $a \le b$, $c \le d$, it follows that $(a, b) = (c, d)$ and $c(n) = 0$. □

For each integer $m \ge 0$ and shell index $k \ge 1$, we define the shell counting function and its cumulative sum by

$$h_m(k) = \#\{n \in S_k : c(n) = m\}, \qquad H_m(N) = \sum_{k=1}^{N} h_m(k), \tag{6}$$

where $h_0(k)$ counts balanced integers in S_k (A382306) and $h_1(k)$ counts eccentric integers with discrepancy 1 in S_k (A383115). To determine the asymptotic behavior of $H_m(N)$, we first bound the number of factor pairs $(a, b) \in S_k \times S_{k+m}$ that fail to be the central divisors of their product $n = ab$.

Lemma 4. *Fix an integer $m \ge 0$. For $X \ge 1$, let $\mathcal{P}_m(X)$ denote the set of ordered factor pairs $(a, b) \in \bigsqcup_{k=1}^{\infty}(S_k \times S_{k+m})$ with $a \le b$ and $ab \le X$ such that $(a, b) \ne (d_1(ab), d_2(ab))$. For each $(a, b) \in \mathcal{P}_m(X)$, let $(c, d) = (d_1(ab), d_2(ab))$ be the central divisor pair of $n = ab$, and set*

$$h = (a + b) - (c + d), \qquad t = b - a, \qquad u = d - c.$$

Then the following properties hold:

1. *The map $\Phi(a,b) = (h,t,u)$ is injective on $\mathcal{P}_m(X)$.*

2. *For $m = 0$, the set $\mathcal{P}_0(X)$ is empty. For every integer $m \geq 1$, the integer h satisfies the uniform bound*
$$1 \leq h \leq m^2 + 2m.$$

3. *For $X = (N+1)^2 - 1$, the cardinality of $\mathcal{P}_m(X)$, as well as the number of multi-pair product collisions in $\bigsqcup_{k=1}^{\infty}(S_k \times S_{k+m})$ up to X, is at most $O_m(N)$.*

Proof. To prove part (1), note that since (c,d) is the central divisor pair of $n = ab = cd$ and $(a,b) \neq (c,d)$, we have $a < c \leq \sqrt{n} \leq d < b$. By the strict convexity of the function $x \mapsto x + n/x$ on $(0,\infty)$, the sum $s = a + b$ strictly exceeds $c + d$, so $h = s - (c+d) \geq 1$ is a positive integer. From the algebraic identity $s^2 - t^2 = 4ab = 4cd = (s-h)^2 - u^2$, expanding $(s-h)^2 = s^2 - 2sh + h^2$ gives

$$t^2 - u^2 = 2sh - h^2, \qquad \text{and hence} \qquad s = \frac{t^2 - u^2 + h^2}{2h}. \tag{7}$$

Because $h \geq 1$, Eq. (7) shows that the triple (h,t,u) uniquely determines $s = a + b$, which together with $t = b - a$ uniquely determines $a = (s-t)/2$ and $b = (s+t)/2$. Thus Φ is injective.

To prove part (2), observe that for any $(a,b) \in S_k \times S_{k+m}$, we have $k \leq \sqrt{a} < k+1$ and $k + m \leq \sqrt{b} < k + m + 1$, which implies $0 \leq \sqrt{b} - \sqrt{a} < m + 1$. By the arithmetic-geometric mean inequality $c + d \geq 2\sqrt{cd} = 2\sqrt{ab}$, we obtain

$$1 \leq h = (a+b) - (c+d) \leq a + b - 2\sqrt{ab} = (\sqrt{b} - \sqrt{a})^2 < (m+1)^2. \tag{8}$$

For $m = 0$, the inequality (8) yields $1 \leq h < 1$ (compare (5)), which is impossible and recovers Lemma 3. For $m \geq 1$, since h is an integer strictly less than $(m+1)^2 = m^2 + 2m + 1$, we have $1 \leq h \leq m^2 + 2m$.

To prove part (3), suppose that $(a,b) \in S_k \times S_{k+m}$ satisfies $ab \leq X = (N+1)^2 - 1$. Then $k^2(k+m)^2 \leq (N+1)^2$, which implies $k \leq N^{1/2} + 1$. Consequently, the differences u and t satisfy

$$0 \leq u < t = b - a \leq (k + m + 1)^2 - 1 - k^2 = 2(m+1)k + m^2 + 2m = O_m(N^{1/2}).$$

Since $1 \leq h \leq m^2 + 2m$, the number of valid triples (h,t,u) is at most $(m^2 + 2m)\binom{O_m(N^{1/2})}{2} = O_m(N)$. Finally, if two distinct pairs $(a_1,b_1), (a_2,b_2) \in \bigsqcup_{k=1}^{\infty}(S_k \times S_{k+m})$ have the same product $n \leq X$, at most one of them can equal $(d_1(n), d_2(n))$, while all remaining pairs belong to $\mathcal{P}_m(X)$. Thus product collisions also contribute at most $O_m(N)$. □

Theorem 5. *The cumulative shell counts $H_m(N) = \sum_{k=1}^{N} h_m(k)$ satisfy the following asymptotic formulas:*

1. *For $m = 0$ (A382306), we have*

$$H_0(N) = \sum_{k=1}^{N} h_0(k) = \frac{2}{3}N^{3/2} + O(N). \tag{9}$$

2. *For every fixed integer $m \geq 1$ (A383115 for $m = 1$), we have*

$$H_m(N) = \sum_{k=1}^{N} h_m(k) = \frac{4}{3}N^{3/2} + O_m(N), \tag{10}$$

and combining (9) and (10) gives $\lim_{N\to\infty} H_m(N)/H_0(N) = 2$ for every fixed $m \geq 1$.

Proof. Let $X = (N+1)^2 - 1$ be the largest integer in shell S_N, so that by (6) we have $H_m(N) = \#\{n \leq X : c(n) = m\}$. Define the cutoff index $K = \lfloor \sqrt{N + m^2/4} - m/2 \rfloor = N^{1/2} - m/2 + O_m(N^{-1/2})$.

For part (1), Lemma 3 implies that every integer $n \leq X$ with $c(n) = 0$ is the product of a central divisor pair $(a, b) \in S_k \times S_k$ with $a \leq b$ and $ab \leq X$. For $1 \leq k \leq K-1$, the largest product in $S_k \times S_k$ is $(k^2+2k)^2 < (N+1)^2 - 1 = X$, so every unordered pair in $S_k \times S_k$ satisfies $ab \leq X$. Because $|S_k| = 2k+1$, the number of pairs $(a, b) \in S_k \times S_k$ with $a \leq b$ is $\binom{2k+2}{2} = 2k^2 + 3k + 1$. The boundary shell $k = K$ contributes at most $O(K^2) = O(N)$ pairs, and by Lemma 4(3), at most $O(N)$ pairs share a common product. Summing over $1 \leq k \leq \lfloor N^{1/2} \rfloor$ gives

$$H_0(N) = \sum_{k=1}^{\lfloor N^{1/2} \rfloor} (2k^2 + 3k + 1) + O(N) = \frac{2}{3}N^{3/2} + O(N),$$

which proves (9).

For part (2), let $m \geq 1$ be fixed. Because the intervals S_k and S_{k+m} are disjoint, every pair $(a, b) \in S_k \times S_{k+m}$ automatically satisfies $a < b$. The Cartesian product $S_k \times S_{k+m}$ contains

$$|S_k| \cdot |S_{k+m}| = (2k+1)(2k+2m+1) = 4k^2 + 4(m+1)k + 2m + 1$$

pairs. Every pair in $S_k \times S_{k+m}$ satisfies $ab \leq X$ for $1 \leq k \leq K-1$, whereas the boundary blocks $k \in \{K, K+1\}$ contain at most $O_m(K^2) = O_m(N)$ pairs. By Lemma 4, non-central pairs and multi-pair collisions account for at most $O_m(N)$ elements. Summing over $1 \leq k \leq K$ yields (10):

$$H_m(N) = \sum_{k=1}^{K} \left(4k^2 + 4(m+1)k + 2m + 1\right) + O_m(N) = \frac{4}{3}N^{3/2} + O_m(N). \qquad \square$$

Remark 6. Classifying balanced integers $n \in S_K$ ($K = \lfloor \sqrt{n} \rfloor$) by the parity of $\delta = d_2(n) - d_1(n)$ links $\mathcal{B} = \mathcal{C}_0$ and $\mathcal{C}_1$ directly to the factor-difference sets of Erdős and Rosenfeld [10] and to the trajectories of the quadratic foliation [5]:

1. If $\delta = 2t$ is even ($t \geq 1$; $t = 0$ gives the squares $n = K^2$), setting $m = (d_1(n) + d_2(n))/2$ yields $n = m^2 - t^2 \in S_{m-1}$, so $m = K + 1$ and $n = (K+1)^2 - t^2 = K^2 + 2K - (t^2 - 1)$ (the diagonal leaf R_{t^2-1} in [5]).

2. If $\delta = 2t + 1$ is odd ($t \geq 0$), writing $d_1(n) = M - t$ and $d_2(n) = M + t + 1$ yields $n = M^2 + M - (t^2 + t) \in S_M$, so $M = K$ and $n = K^2 + K - (t^2 + t)$ (the vertical leaf $V_{-(t^2+t)}$ in [5], starting at the square $(t^2 + t)^2$ when $K = t^2 + t$).

Conversely, for all $K \geq t^2$ in the first family (where $2\sqrt{n} > 2K + 1$) and all $K \geq t^2 + t + 1$ in the second (where $2\sqrt{n} > 2K$), the factor pairs $(K + 1 - t,\ K + 1 + t)$ and $(K - t,\ K + t + 1)$ remain central and satisfy $0 < \sqrt{d_2(n)} - \sqrt{d_1(n)} < 1$, forcing $c(n) \in \{0, 1\}$. As $d_1(n)$ traverses each divisor shell S_q ($q \geq t + 1$), runs of $2q + 1 - \delta$ balanced integers alternate with runs of δ integers of discrepancy 1, while on the short initial segment $\lfloor t^2/2 \rfloor \leq K < t^2$ of R_{t^2-1} ($t \geq 2$), intersections with $V_{-(s^2+s)}$ produce integers with multiple small factor differences [10], such as $12 = 3 \cdot 4 = 2 \cdot 6 \in V_0 \cap R_3$ and $40 = 5 \cdot 8 = 4 \cdot 10 \in V_{-2} \cap R_8$.

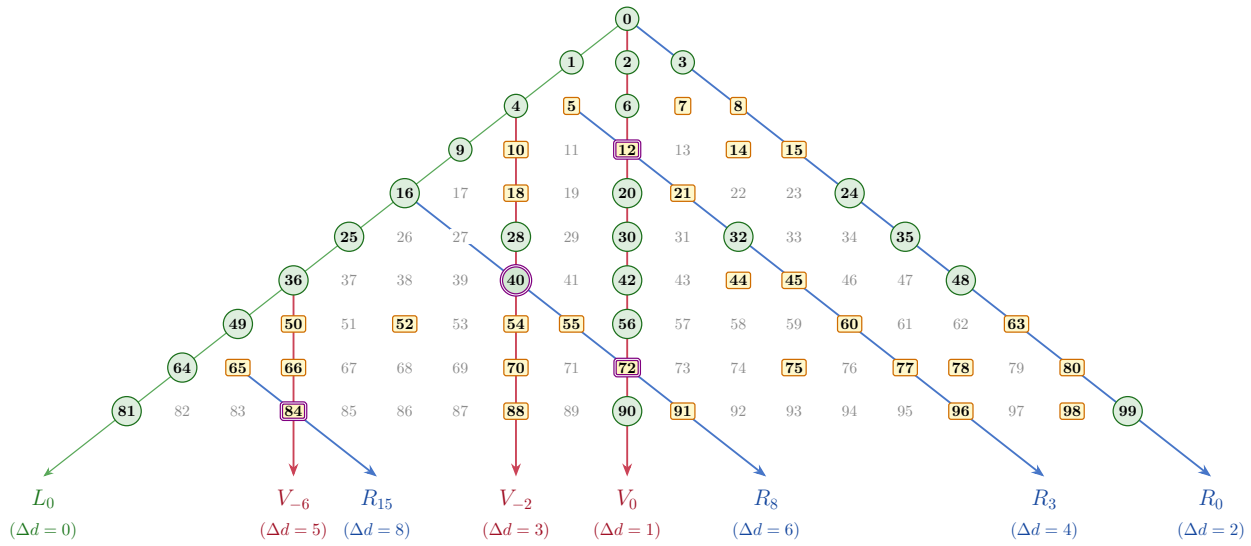


Figure 1: Trajectories of balanced integers ($c(n) = 0$, green circles) and 1-eccentric integers ($c(n) = 1$, amber squares) across shells S_0–S_9 ($0 \leq n \leq 99$) of the quadratic foliation $\mathbb{N}_2$. Opposite-parity central divisor differences $\Delta d = 2t + 1$ trace the holographic vertical leaves $V_{-(t^2+t)}$ (red), where *every* integer has $c(n) \in \{0, 1\}$. Same-parity differences $\Delta d = 2t$ trace the left-to-right diagonal leaves R_{t^2-1} (blue; L_0 for $t = 0$), where $c(n) \in \{0, 1\}$ holds for all $K \geq t^2$. Notice the alternation of contiguous balanced runs and δ-length 1-eccentric runs along V_0 ($\delta = 1$), R_0 ($\delta = 2$), V_{-2} ($\delta = 3$), and R_3 ($\delta = 4$), as well as the Erdős–Rosenfeld multi-difference leaf intersections (double purple rings) at $12 \in V_0 \cap R_3$, $40 \in V_{-2} \cap R_8$, $72 \in V_0 \cap R_8$, and $84 \in V_{-6} \cap R_{15}$.

3 The Minimal Balancing Multiplier $\beta(n)$ and the Square-Balanced Set $\mathcal{L}_2$

We now examine the minimal balancing multiplier $\beta(n)$ defined in (4) (A382286).

3.1 Characterization of Primes, the Prime Counting Function, and Odd Prime Powers

Theorem 7. *For every integer $n \geq 2$, we have $\beta(n) \leq \lfloor\sqrt{n}\rfloor^2$ (A048760). Moreover, an integer $p \geq 2$ is prime (A000040) if and only if $\beta(p) = \lfloor\sqrt{p}\rfloor^2$, or equivalently,*

$$\left\lfloor \frac{\beta(n)}{\lfloor\sqrt{n}\rfloor^2} \right\rfloor = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{if } n \text{ is composite.} \end{cases} \tag{11}$$

Consequently, by (11), the characteristic function of the primes (A010051) is given for $n \geq 2$ by $\chi_{\mathbb{P}}(n) = \lfloor \beta(n)/\lfloor\sqrt{n}\rfloor^2 \rfloor$, and the prime counting function $\pi(x)$ (A000720) satisfies

$$\pi(x) = \sum_{n=2}^{\lfloor x \rfloor} \left\lfloor \frac{\beta(n)}{\lfloor\sqrt{n}\rfloor^2} \right\rfloor \tag{12}$$

for all $x \geq 2$.

Proof. Let $q = \lfloor\sqrt{n}\rfloor$, so that $n \in S_q$ as in (2). Choosing the multiplier $k = q^2$ yields the factorization $q^2 n = q^2 \cdot n$ in which both factors q^2 and n belong to $S_q = [q^2, q^2 + 2q]$. By Lemma 3, we have $c(q^2 n) = 0$, and thus $\beta(n) \leq q^2$.

Suppose first that $n = p$ is prime. Any factorization $kp = ab$ with $1 \leq a \leq b$ must satisfy $p \mid b$, for if $p \mid a$, then $b \geq a \geq p$, which implies $k = ab/p \geq p > q^2$. Writing $b = k_2 p$ and $a = k_1$ with $k_1 k_2 = k$, the condition $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor = m$ requires $m = \lfloor\sqrt{k_2 p}\rfloor \geq \lfloor\sqrt{p}\rfloor = q$ and $k_1 = a \geq m^2 \geq q^2$. Therefore $k = k_1 k_2 \geq q^2$, which proves that $\beta(p) = q^2$.

Conversely, suppose that $n = dv$ is composite with $2 \leq d \leq \sqrt{n} \leq v = n/d$, and set $m = \lfloor\sqrt{v}\rfloor$. Since $d \leq \sqrt{n} \leq v < (m+1)^2$, one readily checks that $d \leq 2m + 1$, so the shell $S_m = [m^2, m^2 + 2m]$ of cardinality $2m + 1$ contains at least one multiple of d. Setting $u_{\min} = \lceil m^2/d \rceil$ places both $u_{\min} d$ and v inside S_m, which by Lemma 3 implies $c(u_{\min} n) = 0$. Since $d \geq 2$, we have

$$\beta(n) \leq u_{\min} = \left\lceil \frac{m^2}{d} \right\rceil \leq \left\lceil \frac{v}{d} \right\rceil \leq \left\lceil \frac{n}{4} \right\rceil < q^2 \tag{13}$$

for all composite $n \geq 4$ (with $n = 4$ satisfying $\beta(4) = 1 < 4$), which establishes (11) and (12). □

Lemma 8 (Target-Shell Characterization of $\beta(n)$)**.** *For any integer $n \geq 2$, let $\mathcal{D}(n) = \{(d, v) \in \mathbb{N}^2 : dv = n \text{ and } 1 \leq d \leq \sqrt{n} \leq v\}$ denote the set of conjugate divisor pairs of n. For each pair $(d, v) \in \mathcal{D}(n)$ and each target shell index $q \geq \lfloor \sqrt{v} \rfloor$, set*

$$u_{\min}(d, q) = \left\lceil \frac{q^2}{d} \right\rceil, \qquad k_{\min}(v, q) = \left\lceil \frac{q^2}{v} \right\rceil,$$

*and call the shell $S_q = [q^2, q^2 + 2q]$ (d, v)-*admissible *if*

$$u_{\min}(d, q)\, d \leq q^2 + 2q \qquad \text{and} \qquad k_{\min}(v, q)\, v \leq q^2 + 2q.$$

Then the minimal balancing multiplier $\beta(n)$ is given by

$$\beta(n) = \min_{\substack{(d,v)\in\mathcal{D}(n) \\ S_q \text{ is } (d,v)\text{-admissible}}} u_{\min}(d, q)\, k_{\min}(v, q), \tag{14}$$

where the minimum may be restricted to target shell indices $q \geq \lfloor \sqrt{v} \rfloor$ satisfying $q^4 < n \lfloor \sqrt{n} \rfloor^2$. (An implementation of this target-shell minimization is given in the OEIS entry A382286.)

Proof. By (4) and Lemma 3, an integer $k \geq 1$ satisfies $c(kn) = 0$ if and only if kn admits a factorization $kn = ab$ with $a, b \in S_q = [q^2, q^2 + 2q]$ for some shell index $q \geq 1$. For any such factorization $kn = ab$, let $d' = \gcd(a, n)$ and $v' = n/d'$. Then $a = u'd'$ and $b = k'v'$ for positive integers u', k' satisfying $u'k' = k$. Relabeling $(d, v) = (\min(d', v'), \max(d', v')) \in \mathcal{D}(n)$ and (u, k_2) accordingly gives a conjugate divisor pair $(d, v) \in \mathcal{D}(n)$ and positive integers u, k_2 with $uk_2 = k$ such that both ud and $k_2 v$ belong to S_q.

Since $ud \in S_q$ and $k_2 v \in S_q$ are equivalent to $q^2 \leq ud \leq q^2 + 2q$ and $q^2 \leq k_2 v \leq q^2 + 2q$ (which requires $q \geq \lfloor \sqrt{v} \rfloor$ because $k_2 \geq 1$), the least positive integers u and k_2 satisfying these inequalities for a fixed pair (d, v) and shell S_q are $u_{\min}(d, q) = \lceil q^2/d \rceil$ and $k_{\min}(v, q) = \lceil q^2/v \rceil$, provided that S_q is (d, v)-admissible. Conversely, for any (d, v)-admissible shell S_q, setting $k = u_{\min}(d, q) k_{\min}(v, q)$ places $u_{\min}(d, q)d$ and $k_{\min}(v, q)v$ inside S_q, so Lemma 3 implies $c(kn) = 0$, proving (14). Finally, since $\beta(n) \leq \lfloor \sqrt{n} \rfloor^2$ by Theorem 7 and any (d, v)-admissible shell S_q satisfies $u_{\min}(d, q) k_{\min}(v, q) n = (u_{\min} d)(k_{\min} v) \geq q^4$, only shells with $q^4 < n \lfloor \sqrt{n} \rfloor^2$ can improve upon the upper bound $\lfloor \sqrt{n} \rfloor^2$. □

While $\beta(p^{2m}) = 1$ for every even prime power, the behavior of β on higher odd prime powers p^{2m+1} $(m \geq 1)$ obeys a scale-invariant functional equation that collapses the exponent $2m + 1$ and recovers the underlying prime base p, with the single exception of $2^5 = 32$.

Theorem 9 (Functional Equation of β on Odd Prime Powers)**.** *For every prime p and every integer $m \geq 1$ with $(p, m) \neq (2, 2)$, the minimal balancing multiplier satisfies the functional equation*

$$\beta(p^{2m+1}) = p \cdot \frac{\beta(p)}{\lfloor \sqrt{p} \rfloor^2} = p, \tag{15}$$

whereas for $(p, m) = (2, 2)$ we have $\beta(2^5) = \beta(32) = 1$. In particular, since a prime p is never a square, $n = 2^5 = 32$ is the unique higher odd prime power p^{2m+1} $(m \geq 1)$ for which $\beta(n)$ is a square.

Proof. Let $n = p^{2m+1}$ with $m \geq 1$. Its central conjugate divisor pair is $(d_1, d_2) = (p^m, p^{m+1})$. Choosing the multiplier $k = p$ factors $pn = (p \cdot p^m) \cdot p^{m+1} = p^{m+1} \cdot p^{m+1}$, which is a square and hence balanced, so $\beta(p^{2m+1}) \leq p$.

To show that (15) holds for all $(p, m) \neq (2, 2)$, consider any conjugate divisor pair $(d_i, v_i) = (p^{m-r}, p^{m+1+r})$ of n with $0 \leq r \leq m$. If $r \geq 1$, the target shell of $v_i = p^{m+1+r}$ has index $K_i = \lfloor p^{(m+1+r)/2} \rfloor$, and any balancing multiplier $k = uk'$ for (d_i, v_i) must satisfy $up^{m-r} \geq K_i^2 > (p^{(m+1+r)/2} - 1)^2$, which implies $u \geq p^{1+2r} - 2p^{(2r+1-m)/2} > p$ for all $(p, m) \neq (2, 1)$, while for $(p, m) = (2, 1)$ $(n = 8)$ direct evaluation gives $\beta(8) = 2$.

Thus the minimal balancing multiplier is achieved by the central pair (p^m, p^{m+1}) at $k' = 1$, where $K = \lfloor p^{(m+1)/2} \rfloor$ and

$$\beta(p^{2m+1}) = \left\lceil \frac{K^2}{p^m} \right\rceil = \left\lceil \frac{\lfloor p^{(m+1)/2} \rfloor^2}{p^m} \right\rceil. \tag{16}$$

We distinguish two cases in (16) according to the parity of m:

1. If $m = 2j + 1$ is odd $(j \geq 0)$, then $(m + 1)/2 = j + 1$ is an integer, so $K = p^{j+1}$ and $K^2/p^m = p^{2j+2}/p^{2j+1} = p$ identically for all primes $p \geq 2$.

2. If $m = 2j$ is even $(j \geq 1)$, then $\sqrt{p^{m+1}} = p^j\sqrt{p}$ is irrational. Write $K = \lfloor p^j\sqrt{p} \rfloor = p^j\sqrt{p} - \theta$ with $\theta \in (0, 1)$. Dividing K^2 by p^{2j} gives

$$\frac{K^2}{p^{2j}} = \left(\sqrt{p} - \frac{\theta}{p^j}\right)^2 = p - \frac{2\theta}{p^{j-1/2}} + \frac{\theta^2}{p^{2j}} > p - \frac{2}{p^{j-1/2}}.$$

Since $K^2/p^{2j} < p$, we have $\lceil K^2/p^{2j} \rceil = p$ whenever $2/p^{j-1/2} \leq 1$, which holds for all $p \geq 5$ and $j \geq 1$ (since $2/\sqrt{5} < 1$) and for all $p \in \{2, 3\}$ and $j \geq 2$ (since $2/2^{3/2} = 1/\sqrt{2} < 1$). For the two remaining cases at $j = 1$ $(m = 2)$: when $p = 3$ $(n = 243)$, $K = \lfloor 3\sqrt{3} \rfloor = 5$ gives $\lceil 25/9 \rceil = 3 = p$; when $p = 2$ $(n = 32)$, $K = \lfloor 2\sqrt{2} \rfloor = 2$ gives $\lceil 4/4 \rceil = 1$. □

3.2 The Square-Balanced Set $\mathcal{L}_2$ and Apéry's Constant $\zeta(3)$

Theorem 7 and Theorem 9 show that $\beta(n)$ is a square for every prime p, every even prime power p^{2m}, and $n = 32$, while excluding all other odd prime powers p^{2m+1}. We now determine the global asymptotic density of the sequence of all positive integers n for which $\beta(n)$ is a square.

Definition 10. The *square-balanced set* $\mathcal{L}_2$ (A401528) is the set of positive integers n such that $\beta(n)$ is a square:

$$\mathcal{L}_2 = \{n \geq 1 : \beta(n) = m^2 \text{ for some integer } m \geq 1\}. \tag{17}$$

The initial terms of $\mathcal{L}_2$ (A401528) are

$$1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 16, 17, 19, 20, 23, 24, 25, 28, 29, 30, 31, 32,$$
$$35, 36, 37, 40, 41, 42, 43, 47, 48, 49, 53, 54, 56, 59, 61, 63, 64, 67, 68, \ldots$$

The integers $n \geq 2$ in $\mathcal{L}_2$ partition into the set of primes $\mathbb{P}$ and the composite subset $\mathcal{L}_2^{\text{comp}} = \{n \geq 4 \text{ composite} : \beta(n) = m^2\}$. Within $\mathcal{L}_2^{\text{comp}}$, we define the *square-prime core*

$$\mathcal{L}_2^* = \{n = r^2 p \in \mathcal{L}_2^{\text{comp}} : r \geq 2 \text{ and } p \text{ is prime}\}.$$

We now show that for $p > r^4$, the membership of $n = r^2 p$ in $\mathcal{L}_2^{\text{comp}}$ is governed by a simple divisibility condition on $\lfloor \sqrt{p} \rfloor$.

Lemma 11. *Let $r \geq 2$ be an integer and let $p > r^4$ be a prime with shell index $K_p = \lfloor \sqrt{p} \rfloor$. Then the following statements hold:*

1. *The minimal balancing multiplier of $n = r^2 p$ is achieved uniquely by the conjugate divisor pair $(d, v) = (r^2, p)$ in the target shell S_{K_p}, and is given by*

$$\beta(r^2 p) = \left\lceil \frac{K_p^2}{r^2} \right\rceil. \tag{18}$$

2. *The integer $n = r^2 p$ belongs to $\mathcal{L}_2^{comp}$ if and only if r divides $\lfloor \sqrt{p} \rfloor$, in which case $\beta(r^2 p) = (\lfloor \sqrt{p} \rfloor / r)^2$.*

Proof. To prove part (1), note that since $p > r^4 > r^2$ is prime, every divisor $d \mid r^2 p$ satisfying $d \leq \sqrt{r^2 p} = r\sqrt{p}$ must divide r^2. For the conjugate divisor pair (r^2, p), since $K_p = \lfloor \sqrt{p} \rfloor \geq r^2$, the shell S_{K_p} containing p has cardinality $2K_p + 1 > r^2$, so setting $k_{\min} = 1$ in (14) gives $u_{\min} = \lceil K_p^2 / r^2 \rceil$ and $\beta(r^2 p) \leq \lceil K_p^2 / r^2 \rceil$. On the other hand, for any proper divisor $d_i \mid r^2$ $(1 \leq d_i \leq r^2/2)$, the conjugate factor $v_i = (r^2/d_i)p$ has shell index $K_i = \lfloor \sqrt{v_i} \rfloor \geq \lfloor \sqrt{2p} \rfloor$, so any balancing multiplier arising from (d_i, v_i) is at least

$$\frac{K_i^2}{d_i} > \frac{(\sqrt{r^2/d_i}\, K_p - 1)^2}{d_i} \geq \frac{4K_p^2}{r^2} - 2rK_p > \left\lceil \frac{K_p^2}{r^2} \right\rceil$$

for all $p > r^4$. Thus (18) holds.

To prove part (2), observe from (18) that $\lceil K_p^2 / r^2 \rceil = m^2$ holds if and only if $r^2(m^2 - 1) < K_p^2 \leq r^2 m^2$, which is equivalent to $\sqrt{r^2 m^2 - r^2} < K_p \leq rm$. Since $p > r^4$ implies $K_p \geq r^2$ and hence $m \geq r \geq 2$, the length of the half-open interval $(\sqrt{r^2 m^2 - r^2}, rm]$ is

$$rm - \sqrt{r^2 m^2 - r^2} = \frac{r^2}{rm + \sqrt{r^2 m^2 - r^2}} < \frac{r^2}{rm + r(m-1)} = \frac{r}{2m - 1} < 1.$$

Because the interval $(\sqrt{r^2 m^2 - r^2}, rm]$ has length strictly less than 1 and integer right endpoint rm, the only integer contained in it is $K_p = rm$. Therefore $\beta(r^2 p) = m^2$ holds if and only if $r \mid K_p$. □

Proposition 12. *Let $r \geq 2$ be an integer, and let $n = r^2 p \in S_k$ with $p > r^4$ prime and $k = \lfloor \sqrt{n} \rfloor$. Then $n \in \mathcal{L}_2^*$ if and only if the shell index k satisfies*

$$(k \bmod r^2) \in \{0, 1, \ldots, r-1\}, \tag{19}$$

in which case $\beta(n) = \lfloor k/r^2 \rfloor^2$. Conversely, whenever $(k \bmod r^2) \in \{r, r+1, \ldots, r^2-1\}$, the shell S_k contains no integers of the form $r^2 p \in \mathcal{L}_2^*$.

Proof. By Lemma 11(2), the integer $n = r^2 p$ belongs to $\mathcal{L}_2^*$ if and only if $\lfloor \sqrt{p} \rfloor = rm$ for some integer $m \geq r$, which is equivalent to $rm \leq \sqrt{p} < rm + 1$. Multiplying across by r gives

$$r^2 m \leq r\sqrt{p} = \sqrt{n} < r^2 m + r.$$

Taking integer parts shows that $k = \lfloor \sqrt{n} \rfloor \in \{r^2 m, r^2 m + 1, \ldots, r^2 m + r - 1\}$, which holds if and only if $m = \lfloor k/r^2 \rfloor$ and (19) is satisfied. □

Theorem 13. *Let $L_2(x)$, $L_2^{comp}(x)$, and $L_2^*(x)$ denote the counting functions of $\mathcal{L}_2$, $\mathcal{L}_2^{comp}$, and $\mathcal{L}_2^*$ up to x. As $x \to \infty$, we have the scale-convolution expansion*

$$L_2(x) = \sum_{r=1}^{\lfloor \sqrt{x/2} \rfloor} \frac{1}{r} \pi\Big(\frac{x}{r^2}\Big) + o\big(\pi(x)\big), \tag{20}$$

which yields the limiting densities

$$\lim_{x\to\infty} \frac{L_2^*(x)}{\pi(x)} = \lim_{x\to\infty} \frac{L_2^{comp}(x)}{\pi(x)} = \sum_{r=2}^{\infty} \frac{1}{r^3} = \zeta(3) - 1 \approx 0.2020569032, \tag{21}$$

and consequently, for the full sequence $\mathcal{L}_2$ (A401528),

$$\lim_{x\to\infty} \frac{L_2(x)}{\pi(x)} = \zeta(3) \approx 1.2020569032 \quad (\text{A002117}). \tag{22}$$

Proof. By unique factorization, every integer $n = r^2 p \in \mathcal{L}_2^*$ ($r \geq 2$, p prime) has a unique representation because p is the unique prime dividing n to an odd exponent (with p^{2m+1} for $m \geq 1$ excluded by Theorem 9 except at $n = 32 = 4 \cdot 8$, where 8 is not prime). Thus $L_2^*(x) = \sum_{r=2}^{\lfloor \sqrt{x/2} \rfloor} \#\mathcal{L}_{2,r}^*(x)$, where $\mathcal{L}_{2,r}^*(x) = \{n = r^2 p \leq x : n \in \mathcal{L}_2^{\text{comp}}\}$.

For each fixed integer $r \geq 2$, Lemma 11 implies that for $r^4 < p \leq x/r^2$, we have $r^2 p \in \mathcal{L}_{2,r}^*(x)$ if and only if $\lfloor \sqrt{p} \rfloor \equiv 0 \pmod r$. Partitioning $[1, x/r^2]$ into shells S_K ($K \leq \lfloor \sqrt{x}/r \rfloor$) and applying summation by parts [2, 19] to the prime number theorem across the r residue classes $K \equiv j \pmod r$ shows that each residue class contains $(1/r + o(1))\pi(x/r^2)$ primes:

$$\#\mathcal{L}_{2,r}^*(x) = \frac{1}{r}\pi\left(\frac{x}{r^2}\right)\big(1 + o(1)\big) = \frac{1}{r^3}\pi(x)\big(1 + o(1)\big).$$

For any cutoff $R \geq 2$, Chebyshev's upper bound $\pi(y) \ll y/\log y$ bounds the tail sum $\sum_{r>R} \#\mathcal{L}^*_{2,r}(x)/\pi(x) \leq \sum_{r>R} \pi(x/r^2)/\pi(x) = O(1/R)$ uniformly in x. Letting $x \to \infty$ and then $R \to \infty$ gives $\lim_{x\to\infty} L^*_2(x)/\pi(x) = \sum_{r=2}^{\infty} r^{-3} = \zeta(3) - 1$ [1].

It remains to bound the transient set $\mathcal{T}_2(x) = \{n \leq x : n \in \mathcal{L}_2^{\text{comp}} \setminus \mathcal{L}_2^*\}$. For any $n \in \mathcal{T}_2(x)$: if $\beta(n) = 1$, Eq. (9) bounds the count by $H_0(\lfloor\sqrt{x}\rfloor) = \frac{2}{3}x^{3/4} + O(x^{1/2})$; if $k_{\min} = 1$ with a non-square divisor d, the interval $(\sqrt{d(m^2-1)}, m\sqrt{d}]$ has length strictly less than $\sqrt{d}/(2m-1)$, contributing at most $O(x^{3/4}\log x)$ integers up to x; and if $k_{\min} \geq 2$ or $v = n/d$ is composite, the block count of Theorem 5 bounds the contribution by $O(x^{3/4}\log x) = o(\pi(x))$. Therefore $L_2^{\text{comp}}(x) = (\zeta(3)-1)\pi(x) + O(x^{3/4}\log x)$, establishing (21). Adding $\pi(x)+1$ for the primes ($r = 1$) and $n = 1$ yields the scale-convolution expansion (20) and the limit (22). □

Corollary 14 (Möbius Scale-Convolution Inversion of $L_2(x)$)**.** *Inverting the scale-convolution expansion* (20) *of Theorem 13 with the Dirichlet kernel $\mu(r)/r$ asymptotically retrieves the prime counting function $\pi(x)$ from the counting function $L_2(x)$ of* A401528*:*

$$\sum_{r=1}^{\lfloor\sqrt{x/2}\rfloor} \frac{\mu(r)}{r} L_2\Big(\frac{x}{r^2}\Big) = \pi(x)\big(1 + o(1)\big), \tag{23}$$

$$\lim_{x\to\infty} \frac{\pi(x)}{L_2(x)} = \sum_{r=1}^{\infty} \frac{\mu(r)}{r^3} = \frac{1}{\zeta(3)}.$$

Proof. Since $\sum_{d|m} \frac{\mu(d)}{d}\frac{1}{m/d} = \frac{1}{m}\sum_{d|m}\mu(d) = \delta_{m,1}$, substituting (20) evaluated at x/d^2 into $\sum_{d\geq 1}\frac{\mu(d)}{d}L_2(x/d^2)$ and interchanging summation via Chebyshev's bound $\pi(x/m^2) \ll m^{-2}\pi(x)$ isolates the $m = 1$ term in (23). □

Remark 15. As shown in Table 2, although $L_2^{\text{comp}}(x)/\pi(x) \to \zeta(3) - 1 < 1$ by (21), the $O(x^{3/4}\log x)$ transient term $T_2(x)$ causes composite terms in $\mathcal{L}_2$ to outnumber primes at small scales. Exact computation up to $x = 4 \cdot 10^7$ shows that $\pi(x)$ first equals $L_2^{\text{comp}}(x)$ at the prime $x_{\text{cross}} = 21{,}038{,}557$, where $\pi(x_{\text{cross}}) = L_2^{\text{comp}}(x_{\text{cross}}) = 1{,}332{,}261$, and satisfies $\pi(x) > L_2^{\text{comp}}(x)$ for all $x \geq 21{,}280{,}079$.

4 Intra-Shell Bounds and the Summatory Function of $\beta(n)$

We next establish an intra-shell upper bound for $\beta(n)$ on composite numbers and determine the asymptotic behavior of the summatory function $\sum_{n\leq x}\beta(n)$.

4.1 The $4:1$ Intra-Shell Gap

Let $\pi^{(2)}(k) = \pi((k+1)^2) - \pi(k^2)$ denote the number of primes in shell S_k (A014085). While unconditional short-interval prime theorems currently guarantee primes in intervals of length

x	$\pi(x)$	$L_2^{\mathrm{comp}}(x)$	$L_2^{\mathrm{comp}}(x)/\pi(x)$	$L_2^*(x)/\pi(x)$	$T_2(x)/\pi(x)$
$5 \cdot 10^4$	5,133	7,229	1.408338	0.299240	1.109098
10^5	9,592	13,214	1.377606	0.289199	1.088407
$5 \cdot 10^5$	41,538	52,933	1.274327	0.266960	1.007367
10^6	78,498	96,022	1.223241	0.259675	0.963566
$2 \cdot 10^6$	148,933	173,875	1.167471	0.253946	0.913525
$5 \cdot 10^6$	348,513	382,995	1.098940	0.246938	0.852003
10^7	664,579	696,924	1.048670	0.242999	0.805671
$1.5 \cdot 10^7$	970,704	991,860	1.021794	0.240476	0.781319
$2 \cdot 10^7$	1,270,607	1,275,539	1.003882	0.239243	0.764638
$2.5 \cdot 10^7$	1,565,927	1,549,365	0.989424	0.238035	0.751389
$3 \cdot 10^7$	1,857,859	1,817,303	0.978171	0.237432	0.740738
$4 \cdot 10^7$	2,433,654	2,340,225	0.961610	0.236066	0.725543
∞	—	—	$\zeta(3) - 1 \approx 0.202057$	$\zeta(3) - 1$	0

Table 2: Empirical counts of $\mathcal{L}_2 = \{1\} \sqcup \mathbb{P} \sqcup \mathcal{L}_2^{\mathrm{comp}}$ (A401528) up to $x = 4 \cdot 10^7$, where $T_2(x) = L_2^{\mathrm{comp}}(x) - L_2^*(x)$.

$x^{0.525}$ [3, 16] and bound the exceptional set of shells [6], the sequence $\beta(n)$ exhibits a strict $4:1$ gap between primes and composites inside each shell S_k.

Proposition 16. *For every integer $k \geq 2$ and every composite integer $n \in S_k$, we have*

$$\beta(n) \leq \left\lceil \frac{k^2}{4} \right\rceil. \tag{24}$$

Consequently, by (24), no integer $n \in S_k$ satisfies $\lceil k^2/4 \rceil < \beta(n) < k^2$, and the shell S_k contains at least one prime ($\pi^{(2)}(k) \geq 1$) if and only if $\max_{n \in S_k} \beta(n) > \lceil k^2/4 \rceil$.

Proof. Let $n \in S_k$ be composite with central conjugate divisors $d = d_1(n) \geq 2$ and $v = d_2(n) = n/d \geq d$, and set $q = \lfloor \sqrt{v} \rfloor$. Since $n \leq k^2 + 2k$ and $d \geq 2$, we have $v = n/d \leq (k^2+2k)/2$. Checking even and odd values of k shows that $q = \lfloor \sqrt{v} \rfloor$ satisfies $q^2 \leq (k^2+1)/2$. As in (13), the multiplier $u_{\min} = \lceil q^2/d \rceil$ places both $u_{\min} d$ and v in S_q, so that

$$\beta(n) \leq \left\lceil \frac{q^2}{d} \right\rceil \leq \left\lceil \frac{q^2}{2} \right\rceil \leq \left\lceil \frac{k^2}{4} \right\rceil,$$

which proves (24). Since $\beta(p) = k^2$ for every prime $p \in S_k$ by Theorem 7, the shell S_k contains at least one prime if and only if $\max_{n \in S_k} \beta(n) > \lceil k^2/4 \rceil$. □

4.2 The Summatory Function of $\beta(n)$

Apéry's constant $\zeta(3)$ [1] governs not only the counting function of square values of $\beta(n)$ in Theorem 13, but also the summatory function of $\beta(n)$ over all integers.

Theorem 17. *As $x \to \infty$, the summatory function $B(x) = \sum_{n=2}^{\lfloor x \rfloor} \beta(n)$ (A382286) satisfies*

$$B(x) = \zeta(3) \sum_{p \le x} \lfloor \sqrt{p} \rfloor^2 + o\left(\frac{x^2}{\log x}\right) = \frac{\zeta(3)}{2} \frac{x^2}{\log x}\big(1 + o(1)\big), \tag{25}$$

where the prime terms contribute weight 1, the composite terms contribute weight $\sum_{d=2}^{\infty} d^{-3} = \zeta(3) - 1$, and Möbius scale-inversion recovers the prime sum $\sum_{d=1}^{\lfloor x/2 \rfloor} \frac{\mu(d)}{d} B(x/d) = \sum_{p \le x} \lfloor \sqrt{p} \rfloor^2 + o(x^2/\log x)$.

Proof. First, for any $y \ge 2$, Abel's summation by parts [2, 19] and the prime number theorem $\pi(t) = \frac{t}{\log t} + O\big(\frac{t}{(\log t)^2}\big)$ give

$$\sum_{p \le y} p = y\pi(y) - \int_2^y \pi(t)\, dt = \frac{y^2}{\log y} - \frac{y^2}{2 \log y} + O\left(\frac{y^2}{(\log y)^2}\right) = \frac{y^2}{2 \log y}\big(1 + o(1)\big). \tag{26}$$

Since $0 \le p - \lfloor \sqrt{p} \rfloor^2 < 2\sqrt{p}$ and $\sum_{p \le y} \sqrt{p} \le \sqrt{y}\, \pi(y) = O(y^{3/2}/\log y)$, it follows from (26) that

$$\sum_{p \le x} \lfloor \sqrt{p} \rfloor^2 = \sum_{p \le x} p + O\left(\frac{x^{3/2}}{\log x}\right) = \frac{x^2}{2 \log x}\big(1 + o(1)\big).$$

To evaluate $B(x) = \sum_{n=2}^{\lfloor x \rfloor} \beta(n)$, let $P^+(n)$ denote the greatest prime factor of n, and recall from (13) that every composite $n \ge 4$ satisfies $\beta(n) \le \lceil d_2(n)/d_1(n) \rceil \le n/d_1(n)^2 + 1$. We partition the integers $2 \le n \le x$ into three classes:

1. If $P^+(n) \le x^{1/3}$, then by multiplying prime factors of n greedily one obtains a divisor $D_1 \mid n$ in $[x^{-1/3}\sqrt{n}, \sqrt{n}]$. Hence $d_1(n) \ge D_1 \ge x^{-1/3}\sqrt{n}$ and $\beta(n) \le n/d_1(n)^2 + 1 \le x^{2/3} + 1$, so all such $n \le x$ contribute at most $O(x^{5/3}) = o(x^2/\log x)$.

2. If $P^+(n) = p > x^{1/3}$ and $p \le \sqrt{n}$, then $d_1(n) \ge p > x^{1/3}$ and $\beta(n) \le n/p^2 + 1 \le x^{1/3} + 1$, contributing at most $O(x^{4/3}) = o(x^2/\log x)$.

3. If $P^+(n) = p > \max(x^{1/3}, \sqrt{n})$, then $n = dp$ with $1 \le d = n/p < \sqrt{n} < p$, so $(d, p) = (d_1(n), d_2(n))$ is the central divisor pair of n. The terms with $d > x^{1/6}$ satisfy $\beta(dp) \le x/d^2 + 1 \le x^{2/3} + 1$ and contribute at most $O(x^{5/3})$.

For each fixed integer $d \ge 2$ and every prime $p > \max(x^{1/3}, \sqrt{n}) \ge d^3$ with $dp \le x$, the proof of Lemma 11(1) gives

$$\beta(dp) = \left\lceil \frac{\lfloor \sqrt{p} \rfloor^2}{d} \right\rceil = \frac{p}{d} + O\left(\frac{\sqrt{p}}{d} + 1\right). \tag{27}$$

Summing (27) over $p \le x/d$ via (26) with $y = x/d$ yields, for each fixed $d \ge 1$,

$$\sum_{\substack{p \le x/d \\ p > x^{1/3}}} \beta(dp) = \frac{1}{d} \sum_{p \le x/d} p + O\left(\frac{x^{3/2}}{d^{5/2} \log x}\right) = \frac{1}{d^3} \frac{x^2}{2 \log x}\big(1 + o_d(1)\big).$$

For any fixed cutoff $D \geq 2$, Chebyshev's bound $\pi(y) \ll y/\log y$ implies that the tail over $D < d \leq x^{1/6}$ satisfies

$$\sum_{D<d\leq x^{1/6}} \sum_{p\leq x/d} \frac{p}{d} \ll \sum_{D<d\leq x^{1/6}} \frac{1}{d} \frac{(x/d)^2}{\log(x/d)} \ll \frac{x^2}{\log x} \sum_{d>D} \frac{1}{d^3} = O\left(\frac{1}{D^2} \frac{x^2}{\log x}\right).$$

Summing over $1 \leq d \leq D$, letting $x \to \infty$, and then letting $D \to \infty$ gives $B(x) = \left(\sum_{d=1}^{\infty} d^{-3}\right) \frac{x^2}{2\log x}(1+o(1)) = \frac{\zeta(3)}{2} \frac{x^2}{\log x}(1+o(1))$, proving (25). Finally, convolving $B(x/d)$ with $\mu(d)/d$ isolates the $d = 1$ prime sum. □

5 Higher Moments of $\beta(n)$, Generalized Power-Balanced Sets, and Odd Zeta Values $\zeta(2s+1)$

Whereas the values of the Riemann zeta function $\zeta(s) = \sum_{r=1}^{\infty} r^{-s}$ at even positive integers $s = 2k$ are rational multiples of π^{2k} and appear classically as the reciprocal densities of $2k$-free integers, the odd zeta values $\zeta(2s+1)$ for $s \geq 1$ (A002117 for $\zeta(3)$, A013663 for $\zeta(5)$, A013665 for $\zeta(7)$) remain central objects of study in arithmetic and Diophantine approximation [1, 4, 7, 8, 11, 18, 20, 21]. We now show that the emergence of Apéry's constant $\zeta(3)$ in (22) and (25) is the $s = 1$ case of a general hierarchy: both the s-th moments of the shell-normalized balancing multiplier and the $2s$-power resonant sets generate all odd zeta values $\zeta(2s+1)$ for every integer $s \geq 1$.

5.1 Moments of the Shell-Normalized Balancing Multiplier and $\zeta(2s+1)$

Recall from (11) and (12) in Theorem 7 that the ratio $\beta(n)/\lfloor\sqrt{n}\rfloor^2$ lies in $(0, 1]$ for all $n \geq 2$ and equals 1 if and only if n is prime.

Definition 18. For each integer $n \geq 2$, the *shell-normalized balancing multiplier* is the rational number

$$\widetilde{\beta}(n) = \frac{\beta(n)}{\lfloor\sqrt{n}\rfloor^2} \in (0, 1]. \tag{28}$$

For each integer $s \geq 1$ and real $x \geq 2$, we define the s-th normalized moment sum and the s-th weighted summatory function by

$$M_s(x) = \sum_{n=2}^{\lfloor x\rfloor} \widetilde{\beta}(n)^s = \sum_{n=2}^{\lfloor x\rfloor} \left(\frac{\beta(n)}{\lfloor\sqrt{n}\rfloor^2}\right)^s, \qquad B^{(s)}(x) = \sum_{n=2}^{\lfloor x\rfloor} \frac{\beta(n)^s}{n^{s-1}}. \tag{29}$$

Because $\widetilde{\beta}(n) \in (0, 1]$ with $\widetilde{\beta}(n) = 1$ if and only if n is prime, decomposing $\widetilde{\beta}(n)^s = \lfloor\widetilde{\beta}(n)^s\rfloor + \{\widetilde{\beta}(n)^s\}$ into its integer part and fractional part $\{y\} = y - \lfloor y\rfloor$ separates the prime contribution from the composite contribution for every $s \geq 1$.

Theorem 19 (The Odd Riemann Zeta Moment Hierarchy of $\beta(n)$)**.** *For every fixed integer $s \geq 1$, the following statements hold:*

1. Integer part and prime counting function: *Generalizing (12), for all $x \geq 2$ the integer parts of $\widetilde{\beta}(n)^s$ sum identically to the prime counting function $\pi(x)$ (A000720):*

$$\sum_{n=2}^{\lfloor x \rfloor} \left\lfloor \widetilde{\beta}(n)^s \right\rfloor = \pi(x).$$

2. Fractional part and odd zeta excess: *As $x \to \infty$, the fractional parts of $\widetilde{\beta}(n)^s$ satisfy the scale-convolution expansion*

$$\sum_{n=2}^{\lfloor x \rfloor} \left\{ \widetilde{\beta}(n)^s \right\} = \sum_{\substack{n=4 \\ n \text{ comp}}}^{\lfloor x \rfloor} \left(\frac{\beta(n)}{\lfloor \sqrt{n} \rfloor^2} \right)^s = \sum_{d=2}^{\lfloor \sqrt{x} \rfloor} \frac{1}{d^{2s}} \, \pi\Big(\frac{x}{d}\Big) + o_s\big(\pi(x)\big), \tag{30}$$

   which yields the limiting ratio

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{n=2}^{\lfloor x \rfloor} \left\{ \widetilde{\beta}(n)^s \right\} = \sum_{d=2}^{\infty} \frac{1}{d^{2s+1}} = \zeta(2s+1) - 1.$$

3. Full s-th moments and Möbius inversion: *Consequently, as $x \to \infty$,*

$$\lim_{x \to \infty} \frac{M_s(x)}{\pi(x)} = \sum_{d=1}^{\infty} \frac{1}{d^{2s+1}} = \zeta(2s+1), \qquad B^{(s)}(x) = \frac{\zeta(2s+1)}{2} \frac{x^2}{\log x} \big(1 + o_s(1)\big), \tag{31}$$

   and Möbius scale-inversion with the Dirichlet kernel $\mu(d)/d^{2s}$ recovers $\pi(x)$ and $1/\zeta(2s+1)$:

$$\sum_{d=1}^{\lfloor x/2 \rfloor} \frac{\mu(d)}{d^{2s}} \, M_s\Big(\frac{x}{d}\Big) = \pi(x)\big(1 + o_s(1)\big), \tag{32}$$

$$\lim_{x \to \infty} \frac{\pi(x)}{M_s(x)} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{2s+1}} = \frac{1}{\zeta(2s+1)}.$$

Proof. To prove part (1), note that by Theorem 7 and (24), the normalized multiplier (28) satisfies $\widetilde{\beta}(p) = 1$ for every prime $p \geq 2$ and $0 < \widetilde{\beta}(n) \leq \lceil k^2/4 \rceil / k^2 < 1$ for every composite $n \geq 4$ ($k = \lfloor \sqrt{n} \rfloor$). Raising $\widetilde{\beta}(n)$ to any positive integer power $s \geq 1$ preserves $\widetilde{\beta}(p)^s = 1$ on primes and $0 < \widetilde{\beta}(n)^s < 1$ on composites. Thus $\lfloor \widetilde{\beta}(n)^s \rfloor = \chi_{\mathbb{P}}(n)$ and $\{\widetilde{\beta}(n)^s\} = (1 - \chi_{\mathbb{P}}(n))\widetilde{\beta}(n)^s$ for all $n \geq 2$, which immediately gives $\sum_{n=2}^{\lfloor x \rfloor} \lfloor \widetilde{\beta}(n)^s \rfloor = \pi(x)$.

To prove part (2), we evaluate $\sum_{n\le x,\, n \text{ comp}} \widetilde{\beta}(n)^s$ by partitioning composite integers $n \le x$ according to their central conjugate divisors $d = d_1(n) \ge 2$ and $v = d_2(n) = n/d \ge d$. Fix an integer $d \ge 2$ and consider first the integers $n = dp \le x$ where $p > d^3$ is prime. By (27), the central divisor pair (d, p) minimizes the balancing multiplier at $k_{\min} = 1$, giving $\beta(dp) = p/d + O(\sqrt{p}/d + 1)$. Since $\lfloor\sqrt{dp}\rfloor = \sqrt{dp} + O(1)$, the denominator satisfies $\lfloor\sqrt{dp}\rfloor^2 = dp + O(\sqrt{dp})$. Dividing $\beta(dp)$ by $\lfloor\sqrt{dp}\rfloor^2$ yields

$$\widetilde{\beta}(dp) = \frac{\beta(dp)}{\lfloor\sqrt{dp}\rfloor^2} = \frac{p/d + O(\sqrt{p}/d)}{dp + O(\sqrt{dp})} = \frac{1}{d^2}\Big(1 + O\big(p^{-1/2}\big)\Big).$$

Raising both sides to the s-th power gives

$$\widetilde{\beta}(dp)^s = \frac{1}{d^{2s}}\Big(1 + O_s\big(p^{-1/2}\big)\Big).$$

Summing over all primes $d^3 < p \le x/d$ and using the prime number theorem $\pi(x/d) = \frac{1}{d}\pi(x)(1 + o(1))$ together with $\sum_{p\le x/d} p^{-1/2} = O((x/d)^{1/2}/\log(x/d)) = o(\pi(x/d))$ yields

$$\sum_{d^3<p\le x/d} \widetilde{\beta}(dp)^s = \frac{1}{d^{2s}}\,\pi\Big(\frac{x}{d}\Big) + O_s\left(\frac{x^{1/2}}{d^{2s+1/2}\log x}\right) = \frac{1}{d^{2s+1}}\,\pi(x)\big(1 + o_s(1)\big).$$

For any fixed cutoff $D \ge 2$, the bounds (13) and (24) imply that $\widetilde{\beta}(n)^s \ll_s d_1(n)^{-2s}$ uniformly for all composite n. Consequently, the tail over $d_1(n) > D$ is bounded by $\sum_{d>D} d^{-2s}(x/d) = O(xD^{-2s}/s)$, while for each fixed $d \le D$, the contribution of composite $n = dv \le x$ for which $v = n/d$ is itself composite (so that n has at least two prime factors besides d, with $d_1(v) \ge d$ and $\widetilde{\beta}(dv) \ll (d \cdot d_1(v))^{-2}$) or $n = p^{2m+1}$ is a higher prime power (where $\widetilde{\beta}(p^{2m+1}) \sim p^{-2m}$ by (15)) is at most $O_D(x/(\log x)^2 + x^{3/4}\log x) = o(\pi(x))$ [9, 19]. Letting $x \to \infty$ and then $D \to \infty$ establishes (30) and part (2).

To prove part (3), adding the prime contribution $\sum_{p\le x} \widetilde{\beta}(p)^s = \pi(x)$ ($d = 1$) to (30) gives the first relation in (31):

$$M_s(x) = \sum_{d=1}^{\lfloor\sqrt{x}\rfloor} \frac{1}{d^{2s}}\,\pi\Big(\frac{x}{d}\Big) + o_s\big(\pi(x)\big) = \zeta(2s+1)\,\pi(x)\big(1 + o_s(1)\big).$$

Similarly, for the weighted summatory function $B^{(s)}(x)$ in (29), on primes ($d = 1$) we have $\beta(p)^s/p^{s-1} = \lfloor\sqrt{p}\rfloor^{2s}/p^{s-1} = p + O_s(\sqrt{p})$, while on $n = dp$ ($d \ge 2$, $p > d^3$ prime) we have

$$\frac{\beta(dp)^s}{(dp)^{s-1}} = \frac{(p/d)^s\big(1 + O_s(p^{-1/2})\big)}{d^{s-1}p^{s-1}} = \frac{p}{d^{2s-1}} + O_s\left(\frac{\sqrt{p}}{d^{2s-1}}\right).$$

Summing over $d^3 < p \le x/d$ via (26) with $y = x/d$ gives $\frac{1}{d^{2s-1}}\frac{(x/d)^2}{2\log x}(1+o(1)) = \frac{1}{d^{2s+1}}\frac{x^2}{2\log x}(1+o(1))$, and summing over all $d \ge 1$ establishes the second relation in (31). Finally, convolving $M_s(x/d)$ with $\mu(d)/d^{2s}$ uses the Dirichlet identity $\sum_{d|m}\frac{\mu(d)}{d^{2s}}\frac{1}{(m/d)^{2s}} = \frac{1}{m^{2s}}\sum_{d|m}\mu(d) = \delta_{m,1}$ to isolate the $m = 1$ term in (32). □

5.2 Generalized $2s$-Power Resonant Sets and Empirical Verification

In addition to appearing in the s-th moments of $\widetilde{\beta}(n)$ over all integers, the odd zeta values $\zeta(2s+1)$ also govern the counting functions of higher-power generalizations of the square-prime core $\mathcal{L}_2^*$ from Subsection 3.2.

Proposition 20 (Scale-Index Recovery and $2s$-Power Resonant Sets)**.** *For each integer $n = r^2p \in \mathcal{L}_2^* \sqcup \mathbb{P}$ ($r \geq 1$, $p > r^4$ prime when $r \geq 2$), the scale index r is recovered directly from n and $\beta(n)$ via*

$$r = \left\lfloor \left(\frac{n}{\beta(n)} \right)^{1/4} \right\rfloor. \tag{33}$$

For each integer $s \geq 1$, define the s-th order resonant balanced set

$$\mathcal{L}_2^{*(s)} = \big\{ N = r^{2s}p : r \geq 1,\ p > r^4 \text{ prime, and } \beta(r^2p) \text{ is a square} \big\}.$$

Then the counting function $L_2^{*(s)}(x) = \#\{N \leq x : N \in \mathcal{L}_2^{*(s)}\}$ satisfies the scale-convolution expansion

$$L_2^{*(s)}(x) = \sum_{r=1}^{\lfloor (x/2)^{1/(2s)} \rfloor} \frac{1}{r}\, \pi\Big(\frac{x}{r^{2s}}\Big) + o_s\big(\pi(x)\big),$$

and consequently

$$\lim_{x\to\infty} \frac{L_2^{*(s)}(x)}{\pi(x)} = \sum_{r=1}^{\infty} \frac{1}{r^{2s+1}} = \zeta(2s+1), \tag{34}$$

$$\sum_{r=1}^{\lfloor (x/2)^{1/(2s)} \rfloor} \frac{\mu(r)}{r}\, L_2^{*(s)}\Big(\frac{x}{r^{2s}}\Big) = \pi(x)\big(1 + o_s(1)\big).$$

Proof. For $n = r^2p$ with $p > r^4$ prime and $\beta(r^2p) = (\lfloor \sqrt{p} \rfloor / r)^2$ (see (18) and Lemma 11(2)), write $\sqrt{p} = K_p + \theta$ with $K_p = \lfloor \sqrt{p} \rfloor \geq r^2 + 1$ and $\theta \in (0,1)$. Then

$$r^4 < \frac{n}{\beta(n)} = r^4 \frac{p}{K_p^2} = r^4 \left(1 + \frac{\theta}{K_p}\right)^2 < r^4 \left(1 + \frac{1}{r^2}\right)^2 < (r+1)^4,$$

so taking fourth roots and integer parts yields (33).

For each fixed $r \geq 1$, an integer $N = r^{2s}p \leq x$ ($p > r^4$ prime) belongs to $\mathcal{L}_2^{*(s)}$ if and only if $p \leq x/r^{2s}$ and $r \mid \lfloor \sqrt{p} \rfloor$ (by Lemma 11(2)). As shown in the proof of Theorem 13, the condition $\lfloor \sqrt{p} \rfloor \equiv 0 \pmod{r}$ selects $(1/r + o(1))\pi(x/r^{2s})$ primes up to x/r^{2s}. Since $\pi(x/r^{2s}) = r^{-2s}\pi(x)(1 + o(1))$, each fixed $r \geq 1$ contributes

$$\frac{1}{r}\, \pi\Big(\frac{x}{r^{2s}}\Big)\big(1 + o(1)\big) = \frac{1}{r^{2s+1}}\, \pi(x)\big(1 + o(1)\big).$$

Summing over $r \geq 1$ and applying Chebyshev's tail bound as in Theorem 13 and (23) yields (34) and its Möbius inversion. □

Table 3 reports the empirical recovery of the odd Riemann zeta values $\zeta(2s+1)$ for $1 \le s \le 7$ from $\beta(n)$ up to $X = 5 \cdot 10^5$.

Order s	Odd Zeta $\zeta(2s+1)$	Exact Value $\sum_{r=1}^{\infty} r^{-(2s+1)}$	Raw Ratio $M_s(X)/\pi(X)$	Core Ratio $M_s^{\mathrm{core}}(X)/\pi(X)$	PNT-Corrected $M_s^{\mathrm{PNT}}(X)/\pi(X)$	Relative Error
1	$\zeta(3)$ (A002117)	1.2020569031	1.2206850242	1.2177614202	1.2014763088	0.048300%
2	$\zeta(5)$ (A013663)	1.0369277551	1.0393845094	1.0393567313	1.0370216001	0.009050%
3	$\zeta(7)$ (A013665)	1.0083492774	1.0088558787	1.0088504670	1.0083611971	0.001182%
4	$\zeta(9)$ (A013667)	1.0020083928	1.0021255377	1.0021235592	1.0020090545	0.000066%
5	$\zeta(11)$ (A013669)	1.0004941886	1.0005225554	1.0005216985	1.0004938287	0.000036%
6	$\zeta(13)$ (A013671)	1.0001227133	1.0001297880	1.0001293879	1.0001224983	0.000021%
7	$\zeta(15)$ (A013673)	1.0000305882	1.0000324125	1.0000322188	1.0000305044	0.000008%

Table 3: Empirical recovery of the odd Riemann zeta values $\zeta(2s+1)$ $(1 \le s \le 7)$ from the minimal balancing multiplier $\beta(n)$ (A382286) evaluated via (14) over $2 \le n \le X = 5 \cdot 10^5$, where $\pi(X) = 41{,}538$. For each order s, the table compares three estimators derived from (29) and Theorem 19: (i) the unfiltered raw moment ratio $M_s(X)/\pi(X) = \pi(X)^{-1} \sum_{n=2}^{X} (\beta(n)/\lfloor\sqrt{n}\rfloor^2)^s$; (ii) the dominant asymptotic core ratio $M_s^{\mathrm{core}}(X)/\pi(X)$ restricted to primes $n = p$ and composite terms $n = dp$ with $p > d^3$ prime, which strips out the $O_s(X^{3/4} \log X)$ small-shell transient; and (iii) the Prime-Number-Theorem log-corrected estimator $M_s^{\mathrm{PNT}}(X)/\pi(X)$, which weights each composite scale $n = dp$ $(p > d^3)$ by the finite-X logarithmic factor $w_{\mathrm{PNT}}(d) = \frac{\log(X/d)}{\log X}$ (canceling the finite-scale drift $\frac{\pi(X/d)}{\pi(X)} \approx \frac{1}{d}\frac{\log X}{\log(X/d)}$) together with the first-order shell-ceiling factor $\left(\frac{\lfloor\sqrt{p}\rfloor^2}{d\,\beta(dp)}\right)^s$. The final column reports the relative percentage error $|M_s^{\mathrm{PNT}}(X)/\pi(X) - \zeta(2s+1)|/\zeta(2s+1)$.

6 The Two-Parameter Foliated Hierarchy $\beta_m^{(r)}(n)$ and the Unification of Even and Odd Zeta Values

In Sections 2–5, the shell discrepancy $c(n)$ in (3) (A381054) and the minimal balancing multiplier $\beta(n)$ in (4) (A382286) were studied relative to the quadratic shells $S_k = [k^2, (k+1)^2 - 1]$ of (2) at target discrepancy $m = 0$. Within the geometric framework of *Geometry of Integers* [5], the quadratic shells $S_{k,2} = S_k$ form the $r = 2$ baseline of an infinite hierarchy of *r-degree integer foliations* $\mathbb{N}_r$ $(r \ge 2)$, in which the non-negative integers are partitioned into horizontal r-th power shells

$$S_{k,r} = \{n \in \mathbb{N} : \lfloor n^{1/r} \rfloor = k\} = [k^r,\ (k+1)^r - 1] \qquad (k \ge 1,\ r \ge 2) \tag{35}$$

of odd cardinality $|S_{k,r}| = (k+1)^r - k^r$ centered along the median spine $V_{0,r}(k) = \big((k+1)^r + k^r - 1\big)/2$.

In this section, we extend our results simultaneously across both natural parameters of the foliation [5]: the *target shell discrepancy* $m \ge 0$ and the *foliation degree* $r \ge 2$. We

prove that while the s-th normalized moments of $\beta_m^{(r)}(n)$ generate the odd Riemann zeta values $\zeta(2s+1)$ universally across all degrees $r \geq 2$ and discrepancies $m \geq 0$, the r-th power multiplier sets $\mathcal{L}_m^{(r)}$ separate into a parity dichotomy: even-degree foliations $\mathbb{N}_{2s}$ ($r = 2s$) generate all odd zeta values $\zeta(2s+1)$, whereas odd-degree foliations $\mathbb{N}_{2s-1}$ ($r = 2s-1$) generate all even zeta values $\zeta(2s) = (-1)^{s+1}\frac{B_{2s}(2\pi)^{2s}}{2(2s)!}$.

6.1 Definitions and Well-Definedness of $c^{(r)}(n)$ and $\beta_m^{(r)}(n)$

Definition 21 (The r-Degree Shell Discrepancy and (m,r)-Multiplier)**.** Fix an integer foliation degree $r \geq 2$ and a target discrepancy $m \geq 0$. For any positive integer $n \geq 1$ with central conjugate divisors $d_1(n)$ and $d_2(n)$ given by (1):

1. The *r-degree shell discrepancy* of n is the non-negative integer

$$c^{(r)}(n) = \lfloor d_2(n)^{1/r} \rfloor - \lfloor d_1(n)^{1/r} \rfloor. \tag{36}$$

   We denote by $\mathcal{C}_m^{(r)} = \{n \geq 1 : c^{(r)}(n) = m\}$ the m-discrepancy set in $\mathbb{N}_r$, and call integers in $\mathcal{B}^{(r)} = \mathcal{C}_0^{(r)}$ *r-balanced*.

2. Let $\mathcal{M}_m^{(r)}(n) = \{k \geq 1 : c^{(r)}(kn) = m\}$ denote the set of m-eccentric multipliers of n in degree r. For every $n \geq 1$, $r \geq 2$, and $m \geq 0$, the set $\mathcal{M}_m^{(r)}(n)$ is non-empty and infinite: for every prime $p > \big(2n(m+1)+n^2\big)^r$ with $K_r = \lfloor p^{1/r} \rfloor$, setting $u = \lceil (K_r - m)^r / n \rceil$ and $k = up$ produces $kn = (un)p$ with central divisors $d_1(kn) = un \in S_{K_r - m, r}$ and $d_2(kn) = p \in S_{K_r, r}$ by Lemma 22 below, so $c^{(r)}(kn) = m$. By the well-ordering principle of $\mathbb{N}_{\geq 1}$, we define the *minimal (m,r)-balancing multiplier* by

$$\beta_m^{(r)}(n) = \min\big\{k \geq 1 : c^{(r)}(kn) = m\big\}. \tag{37}$$

   In particular, $\beta_0^{(2)}(n) = \beta(n)$ (A382286) and $c^{(2)}(n) = c(n)$ (A381054) as in (3) and (4).

For $r = 2$ and $m = 0$, Lemma 3 showed that any factor pair $(a,b) \in S_{k,2} \times S_{k,2}$ is automatically the central divisor pair of ab. For $r \geq 3$ or $m \geq 1$, because the shell cardinality $|S_{k,r}| = (k+1)^r - k^r \sim rk^{r-1}$ exceeds $k^{r/2}$ when $r \geq 3$, a factor pair $(a,b) \in S_{K-m,r} \times S_{K,r}$ is central if and only if no divisor of ab lies strictly between a and b. The following lemma establishes a universal sufficient condition that guarantees centrality for all $r \geq 2$ and $m \geq 0$.

Lemma 22 (Prime-Cofactor Centrality Criterion in $\mathbb{N}_r$)**.** *Let $r \geq 2$ and $m \geq 0$ be integers. If p is a prime and A is any positive integer satisfying $1 \leq A \leq p$, then $(A,p) = \big(d_1(Ap), d_2(Ap)\big)$ is the central conjugate divisor pair of $N = Ap$, and*

$$c^{(r)}(Ap) = \lfloor p^{1/r} \rfloor - \lfloor A^{1/r} \rfloor.$$

Proof. Since p is prime, every divisor $D \mid Ap$ is either not divisible by p, in which case $D \mid A$ and hence $D \le A \le \sqrt{Ap}$ (if $A \le p$), or divisible by p, in which case $D \ge p \ge \sqrt{Ap}$. Consequently, no divisor of Ap lies strictly between A and p, so $d_1(Ap) = A$ and $d_2(Ap) = p$. Substituting these into (36) gives $c^{(r)}(Ap) = \lfloor p^{1/r} \rfloor - \lfloor A^{1/r} \rfloor$. □

Remark 23 (Centrality Phase Transition Between $r = 2$ and $r \ge 3$). Comparing Lemma 3 and Lemma 4 with Lemma 22 highlights a structural phase transition between the quadratic foliation $\mathbb{N}_2$ ($r = 2$) and higher-degree foliations $\mathbb{N}_r$ ($r \ge 3$). In $\mathbb{N}_r$, the k-th shell $S_{k,r}$ of (35) has cardinality $|S_{k,r}| = (k+1)^r - k^r = rk^{r-1} + O(k^{r-2})$, so the total number of unordered factor pairs $(a, b) \in S_{k,r} \times S_{k,r}$ ($a \le b$) with product $ab \le X$ ($k \le X^{1/(2r)}$) is

$$\sum_{k=1}^{\lfloor X^{1/(2r)} \rfloor} \binom{|S_{k,r}| + 1}{2} = \frac{r^2}{2(2r-1)} X^{1-\frac{1}{2r}} + O_r\left(X^{1-\frac{1}{r}}\right), \tag{38}$$

while for $m \ge 1$ the number of factor pairs $(a, b) \in S_{k,r} \times S_{k+m,r}$ with $ab \le X$ is asymptotically twice as large, $\frac{r^2}{2r-1} X^{1-1/(2r)}$. At $r = 2$, the pair sum (38) has exponent $3/4$, and every pair $(a, b) \in S_{k,2} \times S_{k,2}$ satisfies $\sqrt{b} - \sqrt{a} < 1$, which by (5) and (8) forces $h = (a+b) - (d_1(ab) + d_2(ab)) = 0$ and ensures that $\mathcal{P}_0(X) = \emptyset$; thus $r = 2$ is the unique foliation degree in which every same-shell factor pair is unconditionally central, yielding the polynomial asymptotics (9) and (10) of Theorem 5. For $r \ge 3$, by contrast, the shell cardinality $|S_{k,r}| \asymp k^{r-1}$ exceeds $k^{r/2}$, so $(\sqrt{b} - \sqrt{a})^2 \asymp k^{r-2}$ is unbounded and same-shell factor pairs $(a, b) \in S_{k,r} \times S_{k,r}$ can admit intermediate divisors of ab or multi-pair collisions (reducing the unweighted count of r-balanced integers $n \le X$ to a short-interval divisor problem in the sense of Ford [12]). Nevertheless, Lemma 22 shows that whenever the upper factor $b = p$ is prime, no divisor can interpose between A and p; hence centrality is restored unconditionally on the prime and prime-cofactor sets $n = p$, $n = dp$, and $n = d^r p$ for every foliation degree $r \ge 2$.

6.2 Exact Prime Law and Dominant Semiprime Formula for $\beta_m^{(r)}(n)$

Using Lemma 22, we obtain the exact value of $\beta_m^{(r)}(p)$ on all primes p and its asymptotic formula on integers of the form $n = dp$ across all degrees $r \ge 2$ and discrepancies $m \ge 0$.

Theorem 24 (Exact Prime Law and Semiprime Formula for $\beta_m^{(r)}$)**.** *Fix integers $r \ge 2$ and $m \ge 0$.*

1. Exact shifted r-th power law on primes: *For every prime p with r-degree shell index $K_r = \lfloor p^{1/r} \rfloor \ge m + 1$, we have*

$$\beta_m^{(r)}(p) = (K_r - m)^r = \left(\lfloor p^{1/r} \rfloor - m\right)^r. \tag{39}$$

In particular, at $m = 0$, every prime $p \ge 2$ satisfies $\beta_0^{(r)}(p) = \lfloor p^{1/r} \rfloor^r$, and for every integer $n \ge 2$,

$$\left\lfloor \frac{\beta_0^{(r)}(n)}{\lfloor n^{1/r} \rfloor^r} \right\rfloor = \chi_{\mathbb{P}}(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{if } n \text{ is composite.} \end{cases}$$

2. Exact formula for $n = dp$: *Let $d \geq 2$ be a fixed integer. For every prime $p > \big(2d(m+1)+d^2\big)^r$ with $K_r = \lfloor p^{1/r} \rfloor$, the minimal (m,r)-balancing multiplier of $n = dp$ is achieved uniquely by the central divisor pair (d,p) at $k_{\min} = 1$, and equals*

$$\beta_m^{(r)}(dp) = \left\lceil \frac{(K_r - m)^r}{d} \right\rceil = \frac{p}{d} + O_{m,r}\left(\frac{p^{1-1/r}}{d} + 1 \right). \tag{40}$$

Proof. To prove part (1), let p be a prime with $K_r = \lfloor p^{1/r} \rfloor \geq m+1$, and set $k = (K_r - m)^r \geq 1$. Since $k = (K_r - m)^r \leq K_r^r \leq p$, Lemma 22 with $A = k$ implies that (k,p) is the central divisor pair of kp and $c^{(r)}(kp) = K_r - (K_r - m) = m$. By (37), we have $\beta_m^{(r)}(p) \leq (K_r - m)^r$. Conversely, suppose $kp = ab$ with $1 \leq a \leq b$ and $\lfloor b^{1/r} \rfloor - \lfloor a^{1/r} \rfloor = m$. If $p \mid a$, then $b \geq a \geq p > K_r^r \geq (K_r - m)^r$, so $k = ab/p \geq p > (K_r - m)^r$. If $p \mid b$, write $b = k_2 p$ and $a = k_1$ with $k_1 k_2 = k$. Then $\lfloor b^{1/r} \rfloor = \lfloor (k_2 p)^{1/r} \rfloor \geq K_r$, which forces $\lfloor a^{1/r} \rfloor = \lfloor b^{1/r} \rfloor - m \geq K_r - m$, so $k \geq k_1 = a \geq (K_r - m)^r$. This establishes (39). At $m = 0$, any composite $n = dv$ $(2 \leq d \leq \sqrt{n} \leq v)$ has $M_r = \lfloor v^{1/r} \rfloor$ and $u_{\min} = \lceil M_r^r / d \rceil \leq \lceil v/d \rceil < \lfloor n^{1/r} \rfloor^r$ (with $u_{\min} d \leq v$ central when v is prime, or via the central pair of n in general), giving $\lfloor \beta_0^{(r)}(n) / \lfloor n^{1/r} \rfloor^r \rfloor = \chi_{\mathbb{P}}(n)$.

To prove part (2), let $n = dp$ with $p > (2d(m+1)+d^2)^r$ prime, so that $K_r = \lfloor p^{1/r} \rfloor > 2d(m+1)+d^2$. Set $u_{\min} = \lceil (K_r - m)^r / d \rceil$ and $A = u_{\min} d$. Since the shell $S_{K_r - m, r}$ has cardinality

$$(K_r - m + 1)^r - (K_r - m)^r \geq r(K_r - m)^{r-1} \geq 2(K_r - m) > d,$$

the multiple $A = u_{\min} d$ lies inside $S_{K_r - m, r}$, so $\lfloor A^{1/r} \rfloor = K_r - m$ and $A < (K_r - m + 1)^r \leq K_r^r \leq p$ (for $m \geq 1$; for $m = 0$, $A = \lceil K_r^r / d \rceil d < K_r^r + d \leq p + d$, and for $p > d^r$, $K_r^r + d - 1 \leq p$). By Lemma 22, (A, p) is the central divisor pair of $u_{\min} dp$ and $c^{(r)}(u_{\min} dp) = K_r - (K_r - m) = m$, so $\beta_m^{(r)}(dp) \leq u_{\min}$. Any proper divisor $d_i \mid d$ $(1 \leq d_i \leq d/2)$ has cofactor $v_i = (d/d_i)p \geq 2p$ with shell index $K_i \geq \lfloor (2p)^{1/r} \rfloor$, requiring a multiplier at least $(K_i - m)^r / d_i \geq \frac{2}{d}(2^{1/r} K_r - m - 1)^r > \lceil (K_r - m)^r / d \rceil$ for $K_r > 2d(m+1) + d^2$. Finally, expanding $(K_r - m)^r = (p^{1/r} - O(m+1))^r = p + O_{m,r}(p^{1-1/r})$ establishes (40). □

6.3 Universal Odd Zeta Moments $\zeta(2s+1)$ Across All Degrees $r \geq 2$ and Discrepancies $m \geq 0$

We first show that for every foliation degree $r \geq 2$ and every target discrepancy $m \geq 0$, the s-th moments of the shell-normalized multiplier (generalizing (28)) generate the odd Riemann zeta values $\zeta(2s+1)$.

Theorem 25 (Universal Odd Zeta Moment Law Across All (m,r))**.** *Fix integers $r \geq 2$, $m \geq 0$, and $s \geq 1$. For each integer $n \geq 2$, define the (m,r)-normalized multiplier*

$$\widetilde{\beta}_m^{(r)}(n) = \frac{\beta_m^{(r)}(n)}{\lfloor n^{1/r} \rfloor^r}.$$

Then as $x \to \infty$, the s-th moment sum $M_{m,s}^{(r)}(x) = \sum_{n=2}^{\lfloor x \rfloor} \big(\widetilde{\beta}_m^{(r)}(n)\big)^s$ and the weighted summatory function $B_m^{(r,s)}(x) = \sum_{n=2}^{\lfloor x \rfloor} \beta_m^{(r)}(n)^s / n^{s-1}$ satisfy

$$\lim_{x\to\infty} \frac{M_{m,s}^{(r)}(x)}{\pi(x)} = \sum_{d=1}^{\infty} \frac{1}{d^{2s+1}} = \zeta(2s+1), \qquad B_m^{(r,s)}(x) = \frac{\zeta(2s+1)}{2} \frac{x^2}{\log x}\big(1 + o_{m,r,s}(1)\big). \quad (41)$$

Proof. By (39), on primes $n = p > (m+1)^r$ $(d = 1)$, we have

$$\widetilde{\beta}_m^{(r)}(p) = \left(1 - \frac{m}{\lfloor p^{1/r} \rfloor}\right)^r = 1 + O_{m,r}\big(p^{-1/r}\big),$$

so summing $\big(\widetilde{\beta}_m^{(r)}(p)\big)^s$ over all primes $p \le x$ yields $\pi(x) + O_{m,r,s}(x^{1-1/r}/\log x) = \pi(x)(1 + o(1))$.

For composite integers $n = dp \le x$ with $d \ge 2$ fixed and $p > (2d(m+1) + d^2)^r$ prime, Eq. (40) and the estimate $\lfloor (dp)^{1/r} \rfloor^r = dp + O_r((dp)^{1-1/r})$ give

$$\widetilde{\beta}_m^{(r)}(dp) = \frac{\beta_m^{(r)}(dp)}{\lfloor (dp)^{1/r} \rfloor^r} = \frac{p/d + O_{m,r}(p^{1-1/r}/d)}{dp + O_r((dp)^{1-1/r})} = \frac{1}{d^2}\Big(1 + O_{m,r}\big(p^{-1/r}\big)\Big).$$

Raising both sides to the s-th power and summing over all primes $p \le x/d$ yields

$$\sum_{p \le x/d} \big(\widetilde{\beta}_m^{(r)}(dp)\big)^s = \frac{1}{d^{2s}}\, \pi\Big(\frac{x}{d}\Big)\, \big(1 + o_{m,r,s}(1)\big) = \frac{1}{d^{2s+1}}\, \pi(x)\big(1 + o_{m,r,s}(1)\big).$$

As in the proof of Theorem 19, integers $n \le x$ with at least two prime factors below $x^{1/2}$ contribute $O(x/(\log x)^2) = o(\pi(x))$, and the tail over $d > D$ is bounded by $O(D^{-2s}\pi(x))$. Summing over all $d \ge 1$ and applying partial summation yields both relations in (41). □

6.4 The r-th Power Multiplier Sets $\mathcal{L}_m^{(r)}$ and the Parity Unification of $\zeta(r+1)$

While the normalized moments of $\beta_m^{(r)}(n)$ always converge to the odd zeta values $\zeta(2s+1)$, a parity bifurcation occurs when we generalize the square-balanced set $\mathcal{L}_2 = \mathcal{L}_0^{(2)}$ in (17) (A401528) to the *r-th power multiplier set* in the r-degree foliation $\mathbb{N}_r$:

$$\mathcal{L}_m^{(r)} = \big\{n \ge 1 : \beta_m^{(r)}(n) = u^r \text{ for some integer } u \ge 1\big\}, \quad (42)$$

with counting function $L_m^{(r)}(x) = \#\{n \le x : n \in \mathcal{L}_m^{(r)}\}$ and r-th-power-prime core

$$\mathcal{L}_m^{*(r)} = \big\{n = d^r p \in \mathcal{L}_m^{(r)} : d \ge 2 \text{ and } p \text{ is prime}\big\}.$$

Lemma 26 (Modular Shell Resonance for $n = d^r p$ in $\mathbb{N}_r$)**.** *Fix integers $r \geq 2$, $m \geq 0$, and $d \geq 2$. For every prime $p > \big(2d^r(m+1) + d^{2r}\big)^r$ with r-degree shell index $K_r = \lfloor p^{1/r} \rfloor$, the minimal (m, r)-balancing multiplier of $n = d^r p$ is*

$$\beta_m^{(r)}(d^r p) = \left\lceil \frac{(K_r - m)^r}{d^r} \right\rceil.$$

Consequently, $\beta_m^{(r)}(d^r p) = u^r$ is a perfect r-th power if and only if

$$K_r = \lfloor p^{1/r} \rfloor \equiv m \pmod{d}, \tag{43}$$

in which case $\beta_m^{(r)}(d^r p) = \big((\lfloor p^{1/r} \rfloor - m)/d\big)^r$. Equivalently, generalizing (19) by writing $m = qd + \bar{m}$ with $\bar{m} = (m \bmod d) \in \{0, 1, \ldots, d-1\}$, the integer $n = d^r p \in S_{k,r}$ ($k = \lfloor n^{1/r} \rfloor$) belongs to $\mathcal{L}_m^{*(r)}$ if and only if its shell index k lies in the modular band

$$(k \bmod d^2) \in \big\{\bar{m}d,\ \bar{m}d + 1,\ \ldots,\ (\bar{m}+1)d - 1\big\}. \tag{44}$$

Proof. Applying (40) with divisor $D = d^r$ yields $\beta_m^{(r)}(d^r p) = \lceil (K_r - m)^r / d^r \rceil$. From (42), this ceiling equals u^r if and only if

$$d^r(u^r - 1) < (K_r - m)^r \leq d^r u^r \quad \Longleftrightarrow \quad d(u^r - 1)^{1/r} < K_r - m \leq du.$$

Since $K_r > 2d^r(m+1) + d^{2r}$ implies $u \geq d^{r-1} + 1 \geq d + 1$, the mean value theorem applied to $f(t) = t^{1/r}$ on $[u^r - 1, u^r]$ bounds the length of the half-open interval $\big(d(u^r-1)^{1/r},\ du\big]$ by

$$du - d(u^r - 1)^{1/r} < \frac{d}{r(u^r - 1)^{(r-1)/r}} < \frac{d}{r(u-1)^{r-1}} \leq \frac{d}{rd^{r-1}} = \frac{1}{rd^{r-2}} \leq \frac{1}{2} < 1$$

for all $r \geq 2$ and $d \geq 2$. Because the interval $\big(d(u^r-1)^{1/r},\ du\big]$ has length strictly less than 1 and integer right endpoint du, the unique integer contained in it is $K_r - m = du$, which proves (43).

Finally, writing $K_r = du + m = d(u+q) + \bar{m}$ gives $d(u+q) + \bar{m} \leq p^{1/r} < d(u+q) + \bar{m} + 1$. Multiplying across by d yields

$$d^2(u+q) + \bar{m}d \leq dp^{1/r} = n^{1/r} < d^2(u+q) + (\bar{m}+1)d,$$

which holds if and only if $k = \lfloor n^{1/r} \rfloor$ satisfies (44). □

Theorem 27 (The Foliated Zeta Unification Theorem for $\mathcal{L}_m^{(r)}$)**.** *For every foliation degree $r \geq 2$ and every target discrepancy $m \geq 0$, the counting functions $L_m^{*(r)}(x)$ and $L_m^{(r)}(x)$ satisfy the scale-convolution expansion*

$$L_m^{(r)}(x) = \sum_{d=1}^{\lfloor (x/2)^{1/r} \rfloor} \frac{1}{d}\, \pi\Big(\frac{x}{d^r}\Big) + o_{m,r}\big(\pi(x)\big), \tag{45}$$

which yields the exact Riemann zeta density at $r+1$:

$$\lim_{x\to\infty}\frac{L_m^{*(r)}(x)}{\pi(x)}=\sum_{d=2}^{\infty}\frac{1}{d^{r+1}}=\zeta(r+1)-1,\qquad \lim_{x\to\infty}\frac{L_m^{(r)}(x)}{\pi(x)}=\sum_{d=1}^{\infty}\frac{1}{d^{r+1}}=\zeta(r+1),\tag{46}$$

together with the Möbius scale-inversion formula

$$\sum_{d=1}^{\lfloor (x/2)^{1/r}\rfloor}\frac{\mu(d)}{d}\,L_m^{(r)}\Big(\frac{x}{d^r}\Big)=\pi(x)\big(1+o_{m,r}(1)\big).$$

Consequently, across the hierarchy of integer foliations $\{\mathbb{N}_r\}_{r\ge 2}$ [5]:

1. Even-Degree Foliations ($\mathbb{N}_{2s}$, $r=2s$ for $s\ge 1$) Generate All Odd Zeta Values:

$$\lim_{x\to\infty}\frac{L_m^{(2s)}(x)}{\pi(x)}=\zeta(2s+1);\tag{47}$$

   for example, $r=2,4,6$ yield $\zeta(3),\zeta(5),\zeta(7)$, respectively.

2. Odd-Degree Foliations ($\mathbb{N}_{2s-1}$, $r=2s-1$ for $s\ge 2$) Generate All Even Zeta Values:

$$\lim_{x\to\infty}\frac{L_m^{(2s-1)}(x)}{\pi(x)}=\zeta(2s)=(-1)^{s+1}\frac{B_{2s}(2\pi)^{2s}}{2(2s)!};\tag{48}$$

   for example, $r=3$ yields $\zeta(4)=\pi^4/90$ and $r=5$ yields $\zeta(6)=\pi^6/945$.

Proof. By (39), every prime $p>(m+1)^r$ satisfies $\beta_m^{(r)}(p)=(K_r-m)^r$, which is a perfect r-th power; thus the primes contribute $\pi(x)+O_{m,r}(1)$ ($d=1$) to $L_m^{(r)}(x)$.

For each fixed $d\ge 2$, an integer $n=d^rp\le x$ corresponds to a prime $p\le x/d^r$, of which there are $\pi(x/d^r)=d^{-r}\pi(x)(1+o(1))$ by the prime number theorem. By (43) of Lemma 26, for $p>(2d^r(m+1)+d^{2r})^r$, we have $d^rp\in\mathcal{L}_m^{*(r)}$ if and only if $\lfloor p^{1/r}\rfloor\equiv m \pmod d$. Partitioning $[1,x/d^r]$ into r-degree shells $S_{K,r}$ as in (35) ($K\le\lfloor x^{1/r}/d\rfloor$) and applying summation by parts across the d residue classes $K\equiv m \pmod d$ shows that a fraction $1/d+o(1)$ of the primes $p\le x/d^r$ satisfy the congruence. Multiplying the scale factor d^{-r} by the modular resonance factor d^{-1} gives the contribution of each fixed $d\ge 2$:

$$\#\big\{n=d^rp\le x : n\in\mathcal{L}_m^{*(r)}\big\}=\frac{1}{d}\,\pi\Big(\frac{x}{d^r}\Big)\big(1+o(1)\big)=\frac{1}{d^{r+1}}\,\pi(x)\big(1+o(1)\big).$$

For the remaining composite integers $n=Dp\le x$ where D is not a perfect r-th power, the condition $\lceil (K_r-m)^r/D\rceil=u^r$ requires $D(u^r-1)^{1/r}<K_r-m\le D^{1/r}u$, an interval of length $<D^{1/r}/(ru^{r-1})<1$ for $u>D^{1/(r(r-1))}$, which contains at most one integer K_r for each $u\le(x/D^2)^{1/r}$. Since each shell $S_{K_r,r}$ has cardinality $O(K_r^{r-1})$, summing over u and non-r-th-power D bounds the non-core contribution by $O_{m,r}(x^{1-1/(2r)}\log x)=o(\pi(x))$. Summing the main terms over all $d\ge 1$ and applying Chebyshev's tail bound $\sum_{d>R}\pi(x/d^r)/\pi(x)=O(R^{-(r-1)})$ establishes (45) and (46), from which (47) and (48) follow immediately upon setting $r=2s$ and $r=2s-1$. □

Foliation Degree r	Shell Interval $S_{k,r}$ in $\mathbb{N}_r$ [5]	r-th Power Multiplier Set $\mathcal{L}_m^{(r)}$	Limiting Density $\lim_{x\to\infty} \frac{L_m^{(r)}(x)}{\pi(x)}$	Normalized s-th Moment $\lim_{x\to\infty} \frac{M_{m,s}^{(r)}(x)}{\pi(x)}$
$r = 2$	$[k^2, (k+1)^2 - 1]$	$\beta_m^{(2)}(n) = u^2$ ($\mathcal{L}_2$, A401528 at $m = 0$)	$\zeta(3) \approx 1.202057$	$\zeta(2s+1)$
$r = 3$	$[k^3, (k+1)^3 - 1]$	$\beta_m^{(3)}(n) = u^3$	$\zeta(4) = \frac{\pi^4}{90}$ ≈ 1.082323	$\zeta(2s+1)$
$r = 4$	$[k^4, (k+1)^4 - 1]$	$\beta_m^{(4)}(n) = u^4$	$\zeta(5) \approx 1.036928$	$\zeta(2s+1)$
$r = 5$	$[k^5, (k+1)^5 - 1]$	$\beta_m^{(5)}(n) = u^5$	$\zeta(6) = \frac{\pi^6}{945}$ ≈ 1.017343	$\zeta(2s+1)$
$r = 6$	$[k^6, (k+1)^6 - 1]$	$\beta_m^{(6)}(n) = u^6$	$\zeta(7) \approx 1.008349$	$\zeta(2s+1)$
$r = 2s$	$[k^{2s}, (k+1)^{2s} - 1]$	$\beta_m^{(2s)}(n) = u^{2s}$	$\zeta(2s+1)$ (Odd Zeta Values)	$\zeta(2s+1)$
$r = 2s-1$	$[k^{2s-1}, (k+1)^{2s-1} - 1]$	$\beta_m^{(2s-1)}(n) = u^{2s-1}$	$\zeta(2s)$ (Even Zeta Values)	$\zeta(2s+1)$

Table 4: Summary of the two-parameter foliated multiplier hierarchy $\beta_m^{(r)}(n)$ across foliation degrees $r \geq 2$ and target discrepancies $m \geq 0$ (see (41) and (46)). While the s-th moments $M_{m,s}^{(r)}(x)/\pi(x)$ converge universally to $\zeta(2s+1)$, the counts $L_m^{(r)}(x)/\pi(x)$ of the r-th power multiplier sets converge to $\zeta(r+1)$, separating even-degree foliations $\mathbb{N}_{2s}$ (odd zeta values $\zeta(2s+1)$) from odd-degree foliations $\mathbb{N}_{2s-1}$ (even zeta values $\zeta(2s)$).

Remark 28 (Analytic-versus-Geometric Duality in the Generation of $\zeta(2s+1)$). The odd Riemann zeta values $\zeta(2s+1) = \sum_{d=1}^{\infty} d^{-(2s+1)}$ emerge along two distinct avenues in Table 4: (i) from the s-th moment sum $M_{m,s}^{(2)}(x) = \sum_{n=2}^{\lfloor x \rfloor} \widetilde{\beta}_m^{(2)}(n)^s$ inside the quadratic foliation $\mathbb{N}_2$ ($r = 2$, see (31) in Theorem 19), and (ii) from the unweighted counting function $L_m^{(2s)}(x)$ of the $2s$-th power multiplier set $\mathcal{L}_m^{(2s)}$ in (42) across the even-degree foliations $\mathbb{N}_{2s}$ ($r = 2s$, see (47) in Theorem 27).

This reflects a duality between *analytic moment weighting* and *geometric shell thinning* in the factorization of the general Dirichlet term $d^{-(2s+1)}$:

1. *Path 1 — Analytic Moment Realization in $\mathbb{N}_2$ ($r = 2$ Fixed, Continuous Weight $\widetilde{\beta}(n)^s$):* In the moment sum $M_{m,s}^{(2)}(x)$, we remain inside the quadratic foliation $\mathbb{N}_2$ and sum over all composite numbers of the form $n = dp \le x$ ($d \ge 2$, p prime). Because $p \le x/d$, the prime number theorem supplies a counting density of $\pi(x/d) \sim \frac{1}{d}\,\pi(x)$. The remaining factor d^{-2s} is supplied analytically by raising the ratio $\widetilde{\beta}_m^{(2)}(dp) = \beta_m^{(2)}(dp)/\lfloor\sqrt{dp}\rfloor^2 \sim (p/d)/(dp) = d^{-2}$ to the s-th power:

$$\underbrace{\frac{1}{d}}_{\substack{\text{prime count}\\ p\le x/d}} \times \underbrace{\left(\frac{1}{d^2}\right)^s}_{\substack{\text{analytic weight}\\ \widetilde{\beta}_m^{(2)}(dp)^s}} = \frac{1}{d^{2s+1}} \quad\Longrightarrow\quad \sum_{d=1}^{\infty} \frac{1}{d^{2s+1}} = \zeta(2s+1).$$

2. *Path 2 — Geometric Counting Realization in $\mathbb{N}_r$ (Unit Indicator Weight):* In the counting function $L_m^{(r)}(x) = \#\{n \le x : \beta_m^{(r)}(n) = u^r\}$, every integer in the set $\mathcal{L}_m^{(r)}$ carries unit weight 1. To achieve an r-th power multiplier u^r in the r-degree foliation $\mathbb{N}_r$ [5], the dominant composite integers have the form $n = d^r p \le x$. Consequently, the prime bound $p \le x/d^r$ thins the prime count geometrically by $\pi(x/d^r) \sim \frac{1}{d^r}\,\pi(x)$, while the modular shell congruence (43) selects a fraction $\frac{1}{d}$ of those primes:

$$\underbrace{\frac{1}{d^r}}_{\substack{\text{geometric scale thinning}\\ p\le x/d^r \text{ in } \mathbb{N}_r}} \times \underbrace{\frac{1}{d}}_{\substack{\text{shell congruence}\\ \lfloor p^{1/r}\rfloor \equiv m \pmod d}} = \frac{1}{d^{r+1}} \quad\Longrightarrow\quad \sum_{d=1}^{\infty} \frac{1}{d^{r+1}} = \zeta(r+1).$$

   When the foliation degree is even ($r = 2s$), $r+1 = 2s+1$ is odd, recovering $\zeta(2s+1)$ in (47) as the asymptotic density of an integer sequence; when the foliation degree is odd ($r = 2s-1$), $r+1 = 2s$ is even, producing the even zeta values $\zeta(2s)$ in (48).

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